Geometric Flows and The Swampland

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The Ricci flow is a geometric flow much like a vector flow, but used to expand or contract geometric spaces to analyze their structure. We review basic mathematical properties of Ricci flow and its analogies with the heat equation, and examine how Ricci flow has been applied to different areas in physics. We focus in particular on recent string theory work that has utilized Ricci flow in the context of the Swampland Program and the Distance Conjecture, which we review. As a particular application, we explore whether such methods can be extended to more general geometries and ask whether Ricci flow can be used to obtain sharper criteria to distinguish the Landscape from the Swampland.

1. Review of Geometric Flows

1.1. What Constitutes a Geometric Flow?

Recall from multivariable calculus that the following operations, gradient and divergence, create vector flows. However, a *geometric flow* takes these concepts, and upgrades them by allowing them to operate on the entire geometric space, or *manifold*, itself. This evolves as either contractions or expansions of the geometry. A geometric flow with some operator L given by $\partial_t u = Lu$ is interpreted as a flow if L satisfies the following:

- A linear operator on states.
- An elliptic operator; meaning that if $Lu = \sum_i a_i(x)\partial^i u$, then for every $x^i \in \mathbb{R}^n$, and every non-zero $\xi \in \mathbb{R}^n$ they obey:$\sum_i a_i(x)\xi^i \neq 0$.

1.2. The Ricci Flow

Above, we defined geometric flows for a PDE, which is on a Euclidean manifold. However they are extended in general. For Ricci flow, the PDE is a tensor equation, and we are on a Riemmanian manifold. We will define the Ricci flow, and show an example of it operating on a topological object in figure 1.

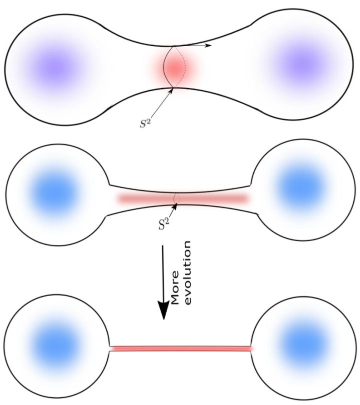

Fig. 1. Ricci flow example.

Definition 1 (The Ricci Flow) *On a smooth and closed manifold $\mathscr{M}$ that is described by a smooth Riemannian metric $g_{\mu\nu}$, we can define the Ricci flow as a geometric flow evolving under a flow-parameter t being related to the curvature described by the Ricci tensor $R_{\mu\nu}$:*

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}. \tag{1}$$

In figure 1, we start with the top image, where red indicates areas of high curvature, and blue are areas of low curvature[1]. If we define our basis vectors along the middle 2-sphere, and evolve using Ricci flow, then the middle shall contract, while the outside remains as two spheres. Imagine a circle at every point along the space. Ricci flow evolves along the circles as $\frac{1}{r}$. Then, if r is large, the evolution is slow, but if r is small, the evolution is fast. Eventually, we come to a point where we can't evolve any further called a *singularity*, meaning we either stop, or perform *surgery*[2].

Other than using Ricci flow to prove the Poincáre conjecture, within the context of physics, it has been used in computing the beta functions for sigma-models via renomalization group flow, used in topology changing processes between black holes and string theory, and may be helpful in general relativity, specifically in computing potential energy of black holes and mass distribution in static metrics. However in this note, we will focus on the role Ricci flow has played in the context of the string theory Swampland program.

[1]It may be easier to think of Ricci flow as the *heat equation*, $\frac{\partial u}{\partial t} = \nabla^2 \mathbf{u}$, by relating $g_{\mu\nu} \propto u$ & $R_{\mu\nu} \propto \nabla^2 \mathbf{u}$, or directly via a conformal metric, $g_{\mu\nu} = exp(p(x,y)) * diag(1,1)$.

[2]Surgery theory is a field of mathematics that allows you to remove singularities, and replace them with *caps*; in our case, *3-balls*.

2. The Swampland, Distance Conjectures, & Incorporating Ricci Flow

2.1. *Introducing the Swampland & Distance Conjectures*

The *Swampland* can be defined as the set of (seemingly) consistent effective quantum field theories that cannot be completed into quantum gravity in the ultraviolet energy regime [4], and in particular cannot arise from string theory. On the other hand, the string theory Landscape describes those effective field theories which are compatible with quantum gravity. An important fundamental question is to understand what kinds of constraints quantum gravity places on low-energy theories, and to find criteria to distinguish the Swampland from the Landscape. One of the criteria that has been identified is called the Distance Conjecture, which states that an effective field theory (parametrized by some massless scalar fields or "moduli") is only valid for a finite range of motion of such scalars, because an infinite tower of massive states becomes exponentially light in the limit of infinite field range. The latter signals the break-down of the effective theory.

There are more examples of distance conjectures, namely on Kähler manifolds [5], anti-de Sitter spacetimes [1], and Calabi Yau manifolds.

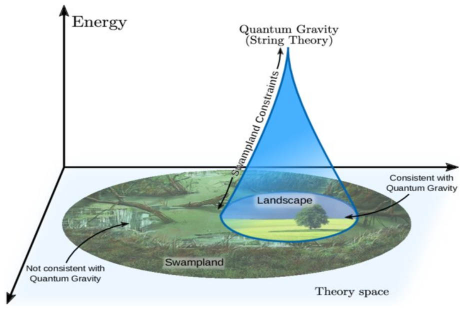

Fig. 2. Going from Swampland to Landscape.

2.2. *Where Ricci Flow Comes into Play*

In [1], Lüst showed that the Ricci flow can be used to derive the distance conjecture for anti-de Sitter spacetimes. This is done by giving time-dependence to the cosmological constant $\Lambda(t)$, plugging the metric into the Ricci flow, and solving for the evolution of the metric. This can then be put into the geodesic along a path Δ_g, giving an explicit distance formula in field space. The associated tower of massive states would be described by,

$$m(t') \sim m(t)e^{-\alpha|\Delta_g(t,t')|} \tag{2}$$

which goes to zero as $\Delta_g \to \infty$, as stated by the distance conjecture.

This example shows that the Ricci flow provides an efficient method for calculating the distance Δ_g in field space, and can be useful for refining the distance conjecture. Thus, we want to examine to what extent Ricci flow can be applied to *general* geometries and constructions, and how to use it to identify additional criteria to distinguish the Landscape from the Swampland.

For example, [5] gave reasonable evidence for a distance conjecture for Kähler moduli after finding discrete symmetries associated with an infinite tower of states. One question is whether one could use Ricci flow to find the same result. Kähler manifolds *do* have a metric, but in terms of *holomorphic* coordinates (complex). This suggests a modification of Ricci flow. Doing so, by a change of coordinates $g_{\mu\nu} \to g_{\alpha\bar{\beta}}$ we find a known flow called *Calabi* flow,

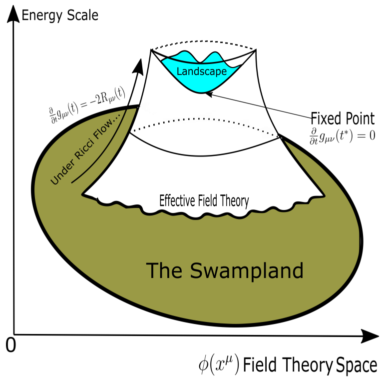

Fig. 3. Geometric interpretation.

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial t} = \frac{\partial R(g)}{\partial z^\alpha \bar{z}^\beta} \tag{3}$$

where z^α are holomorphic coordinates. To proceed we would find the time-dependent parameters in the Kähler metric, evolve under Calabi flow, and follow the same procedure outlined above. Incorporating geometric flows onto the Swampland demands a new interpretation of how to evolve geometries arising from a particular effective field theory, starting at low energies and approaching the quantum gravity regime. A sketch of what this may look like is depicted in figure 3. Both the example and geometric interpretation are ongoing projects.

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