Brans-Dicke Cosmology Model with a Fermionic Field

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Abstract

We investigate Brans-Dicke gravity with a fermionic field by deriving the Friedman-Robertson-Walker equations. The Brans-Dicke Lagrangian is in the usual formal context, but with a cosmological constant of 2Λ. The scalar field ϕ(t) is scaled by a factor of: $\phi(\mathrm{t}) \propto 1/G$. The fermionic field is taken as an effective field theory in the sense it has only viable solutions below the Planck limit. Both fields also have a self interaction potential of V(ϕ) corresponding to the scalar field from Brans-Dicke, and V(Ψ) corresponding to the fermion field. From these, we study the Friedmann corresponding to curvature values, $k = -1, 0, 1$, and whether or not the scalar and/or the fermions have a mass. In the analysis, we find that depending on the choice of the potential's power, we find either inflation or long-time expansion.

1 Preliminary Information

Currently, the best description for a gravitational system is Einstein's general relativity in the form of $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$ with $c = 1$, which will be for the rest of this paper. However as successful this theory has been for over a century now, it is still not the final piece. One of the reasons has been how inefficient it is at explaining, and satisfying Mach's principle [4]. A way of solving this was found in 1961 by Brans and Dicke [5]. This is done by interpreting Newton's constant G as not a constant but as an evolving dynamic scalar field.

The way of doing this is to vary $G \propto \frac{1}{\phi}$ with dimensions ML^{-1}. This helps Mach's principle as well as help satisfy the strong equivalence principle. Having done this, the action comes in the form as

$$S = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\phi(R + 2\Lambda) - \frac{\omega}{\phi}(\partial\phi)^2 - V(\phi) \tag{1}$$

There is also a self-interaction potential that will be determined later on. The third term with the $\partial\phi$ can, and will often be expanded as

$$(\partial\phi)^2 = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{2}$$

This is because a scalar field is invariant under change between the covariant derivative, and its normal partial derivative since a scalar field's Christoffel symbols are always zero. We have also added for convenience the constant 2Λ as later on it will help distinguish between dynamics of universal expansion. We can also see that the scalar field will have a direct relationship on the nature of spacetime curvature. The constant ω is a dimensionless Dicke-coupling constant to the scalar field.

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As $\omega \to \infty$ we recover the original dynamics of general relativity, in the sense that since ω becomes so large, it dominates ϕ and thus its overall contribution to the curvature terms become infinitely small or zero.

In order to describe a femionic field in this context we need to formulate fermions in a curved background. Recall that the usual Dirac equation governing fermions is given by,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{3}$$

To put the equation in a curved background, we need need a connection coefficient. For spinors, this is a connection with respect to a spinor manifold that gives rise to the spinor-bundle connection[8],

$$\begin{aligned} &\nabla_\mu \Psi = \partial_\mu \Psi - \Omega_\mu \Psi \ , \nabla_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} + \Omega_\mu \bar{\Psi} \ , \\ &\Omega_\mu = -\frac{1}{4} g_{\rho\sigma} [\Gamma^\rho_{\mu\delta} - e^\rho_b \partial_\mu e^b_\delta] \gamma^\sigma \gamma^\delta , \end{aligned} \tag{4}$$

which is just the complex conjugate of one or the other.

Ω_μ is what we shall define as the *spin connection*. This is analogous to the regular spacetime connections, formally the christoffel symbols but instead directly to spacetime, it is a connection on a spinor bundle. This is where our spinor field describing fermions live. Also, we shall be very restrictive on language usage for indices. Such as if we see γ^a then this is simply one of the four dirac matrices in flat spacetime (which we shall define as $\eta^{\mu\nu} = diag(1, -1, -1, 1-))$. But if we see γ^μ then this is our gamma matrices in curved spacetime, or the generalized Dirac-Pauli matrices. The transition between flat and curved for our matrices are through tetrad formalism or *vierbein* [9] as $\gamma^\mu = e^\mu_a \gamma^a$.

This will allow us to find an action that allows us to use the variational mechanics in curved spacetime. This action is of the following form,

$$S = \int d^4x \sqrt{-g}\, \frac{i}{2} [\bar{\Psi}\gamma^\mu \nabla_\mu \Psi - \Psi \gamma^\mu \nabla_\mu \bar{\Psi}] - m(\bar{\Psi}\Psi) - V(\bar{\Psi}\Psi) \tag{5}$$

This will give us two equations of motion with respect to $\bar{\Psi}$ and Ψ.

Our metric will be that of the general Friedmann-Robertson-Walker metric,

$$g_{\mu\nu} = dt^2 - a(t)^2 [\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2], \ d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{6}$$

With k having possible values of $-1, 0, 1$ corresponding to a closed 3-sphere, flat, or an open 3-hyperboloid.

2 Worked Out Derivations

This section shall be the main bulk of this paper. This entire chapter is devoted towards the derivation of the equations of motion from our lagrangians previously introduced using variational principle. We shall first start by finding the most basic, but rigorous of them, $\frac{\delta S}{\delta g^{\mu\nu}}$. To do this, we will break the action into two pieces as $S = S_g + S_f$, which denotes the gravitational part the Brans-Dicke action, and the fermionic action respectively. This shall be followed up by finding the equations of motion with respect to ϕ and then Ψ. Afterwards, we turn our attention to describing our stress-energy tensor which comes from our matter lagrangian S_f when we vary it with respect to the metric.

We will then find the metric signatures of G^{00} and G^{11} along with the time-dependent equations of motion from our ϕ and Ψ equations. This will give us the Friedmann equations for this particular system that will allow us to evaluate and see the dynamics of this model. We shall see whether or not we have a fair model of what some may deem as inflation, or dark energy (long-term expansion rates).

2.1 Brans-Dicke Equations

We shall begin by taking a look at the action for Brans-Dicke we previously wrote down.

$$S = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\phi(R+2\Lambda) - \frac{\omega}{\phi}(\partial\phi)^2 - V(\phi) \tag{7}$$

From here, we shall take variation with respect to the action of the form: $\frac{\delta S}{\delta g^{\mu\nu}} = 0$. We take this variation with respect to the contravariant form of the metric in order to have a covariant equation of motion. This is solely based upon convention, as one could do the opposite and have a contravariant form, but with a possible minus sign on some spatial components. We are also going to assume for the remainder of this paper that ϕ has no dependence on $g_{\mu\nu}$. This means that we can treat ϕ as a constant in this first variation.

A first expansion of the variation gives,

$$\delta_g S = \int d^4x\delta(\sqrt{-g})\mathcal{L} + \int d^4x\sqrt{-g}\left[\phi\delta(R-2\Lambda) - \frac{\omega}{\phi}\delta(\partial\phi)^2 - \delta V(\phi)\right] \tag{8}$$

The first portion of the variation is with respect to our tensor density for spacetime integrals $\delta\sqrt{-g}$. To find this quantity, there are two ways, one straight forward using functional formalism recognizing that the quantity $\sqrt{-g}$. Or, we can take a less mathematical approach, and use a relationship from linear algebra. We shall first look at the latter of the two.

Lets take a matrix A, and use the relationship between exponentials and determinants $det(e^A) = e^{Trace(A)}$. Now if we define $B = e^A$, and with a little algebra;

$$B = e^A \to A = ln(A), det(B) = e^{Tr(ln(B))},$$

$$ln(det(B)) = ln(e^{Tr(ln(B))}) = Tr(ln(B)) \tag{9}$$

If we now take the differential of each side,

$$\frac{\partial B}{det(B)} = Tr(\frac{\partial(B)}{B}) = Tr(\partial(B) * B^{-1}) \tag{10}$$

Now, set $B = g_{\mu\nu}$,$B^{-1} = g^{\mu\nu}$ and change the ∂ to δ,

$$\frac{\delta g}{g} = -g_{\mu\nu}\delta g^{\mu\nu} \to \delta g = -g g_{\mu\nu}\delta g^{\mu\nu} \tag{11}$$

Now, after taking the functional derivative of $\sqrt{-g}$ and applying our new expression for ∂g we shall find,

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2\sqrt{-g}}g g_{\mu\nu}\delta g^{\mu\nu} \;\; = -\frac{1}{2\sqrt{-g}}(-g)g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2\sqrt{-g}}(\sqrt{-g})^2 g_{\mu\nu} \tag{12}$$

To which we finally find:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{13}$$

The other method for arriving at this result is using complete functional formalism which can be done in about one line,

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\frac{-g}{\sqrt{-g}}g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}, \tag{14}$$

and if you multiply over the $\delta g^{\mu\nu}$, then you arrive at the same result. This same results can be found in [4]. If we plug this result back into the first part of our integral, we find,

$$\delta_g S = \int d^4x \delta(\sqrt{-g})\mathcal{L} \rightarrow \int d^4x\sqrt{-g}\delta g^{\mu\nu}(-\frac{1}{2}g_{\mu\nu}\mathcal{L})$$

$$= \int d^4x\sqrt{-g}\delta g^{\mu\nu}[(g_{\mu\nu}\Lambda - \frac{1}{2}Rg_{\mu\nu})\phi - g_{\mu\nu}\frac{\omega}{2\phi}(\partial\phi)^2 + \frac{1}{2}g_{\mu\nu}V(\phi)] \tag{15}$$

This gives us the first piece towards finding the equations of motion. From which, we can already see familiar bits and pieces of equations that we know so well; such as the second portion of the Einstein tensor, and the form of the potential. Next, we shall visit the dynamics of what happens when we vary the piece: $\phi\delta(R - 2\Lambda)$. This will lead us down a path that brings us to something called the Palatini approach.

2.2 Variation of the Riemann Tensor

We begin by looking at a section of the Brans-Dicke action;

$$S = \int d^4x\sqrt{-g}[\phi\delta(R - 2\Lambda)] \tag{16}$$

We first begin by recognizing that $2\phi\delta\Lambda$ goes to zero since Λ does not depend upon $g_{\mu\nu}$. Actually, Λ is more of a 'place-holder' to note that we understand there is currently an un-described force of nature pulling apart the seams of the universe on the large distances, formally known as Dark Energy. As a side note, Λ is sometimes solved and set as a constant. For example, in AdS space (Anti-de Sitter), $\Lambda = \frac{6}{L^2}$ in 4 dimensions. So all we have left is is $= \int d^4x\sqrt{-g}\phi\delta R$.

Now, if we expand the Ricci scalar, and remember the formal definition of the Ricci scalar from the Ricci Tensor, we find the following;

$$\begin{aligned}
&R = g^{\mu\nu}R_{\mu\nu}; R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \\
&R^{\lambda}_{\mu\lambda\nu} \overset{LF}{=} \partial_\lambda\Gamma^{\lambda}_{\nu\mu} - \partial_\nu\Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}, \\
&\Gamma^{\lambda}_{\lambda\sigma} \overset{LF}{=} 0 \rightarrow R^{\lambda}_{\mu\lambda\nu} \overset{LF}{=} \partial_\lambda\Gamma^{\lambda}_{\nu\mu} - \partial_\nu\Gamma^{\lambda}_{\lambda\mu}, \\
&R^{\lambda}_{\mu\lambda\nu} = \nabla_\lambda\Gamma^{\lambda}_{\nu\mu} - \nabla_\nu\Gamma^{\lambda}_{\lambda\mu} = R_{\mu\nu}.
\end{aligned} \tag{17}$$

'LF' denotes local inertial frame, which physically means we are looking at a small enough region of spacetime that is flat, or, obeys Newton's laws of motion. For the more mathematically inclined, we take a look at our spacetime manifold M, and find in a region where the metric connections (christoffel symbols) are zero, means that we are in a region of flat spacetime, but the first derivative does not disappear. And since in a local frame, we still have a tensor, we can naturally promote the partial derivative to a covariant one.

Now, if we look at a variation of the Ricci tensor; $\bar{R}_{\mu\nu} = R_{\mu\nu} + \delta R_{\mu\nu}$, the variation once expanded is simply:

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu}) \tag{18}$$

This is the Palatini equation for the Ricci tensor. It's significance is crucial in the understanding of relativity from the variational principle, since it gives us an insight into the dynamics of the Ricci tensor. From here, we plug this back into our original expression,

$$\begin{aligned}
\delta_g S &= \int_M d^4x\sqrt{-g}\phi\delta R = \int_M d^4x\sqrt{-g}\phi(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}) \rightarrow \\
&= \int_M d^4x\sqrt{-g}\phi g^{\mu\nu}\delta R_{\mu\nu} = \int_M d^4x\sqrt{-g}\phi g^{\mu\nu}[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})].
\end{aligned} \tag{19}$$

The first line is the expansion of the Ricci scalar, and the second line is solely the second portion of the expansion since the first is already in a form we enjoy. If we multiply out the $\phi g^{\mu\nu}$ into the covariant derivatives, we find,

$$\nabla_\lambda(\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) = \nabla_\lambda\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} + \phi g^{\mu\nu}\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) \tag{20}$$

And similar for the second term in 2.12. The first term of the RHS (right hand side) goes to zero by virtue of Gauss's theorem. So all we have left is the term, $\nabla_\lambda\phi[g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}]$. And if we finally plug in the variation of the Christoffel symbol: $\delta\Gamma^\lambda_{\mu\lambda} = -\frac{1}{2}g_{\lambda\sigma}\nabla_\mu(\delta g^{\lambda\sigma})$:

$$\begin{aligned} \delta_g S &= \int_M d^4x\sqrt{-g}[\nabla_\nu\phi\nabla_\mu\delta g^{\mu\nu} - \nabla_\lambda\phi g_{\mu\nu}\nabla^\lambda(\delta g^{\mu\nu})] \to \\ &= \int_M d^4x\sqrt{-g}\delta g^{\mu\nu}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\phi. \end{aligned} \tag{21}$$

2.3 Variation of the Scalar Field

Our last piece is the variation of the scalar field. Here, we are going to take the variation of the scalar field potential to be $\delta V(\phi) = 0$. So all we are left with is,

$$\delta_\phi S = \int_M d^4x\sqrt{-g}[-\delta(\frac{\omega}{\phi}(\partial\phi)^2)] = \int_M d^4x\sqrt{-g}[-\frac{\omega}{\phi}(\nabla_\mu\phi\nabla_\nu\phi)\delta g_{\mu\nu}] \tag{22}$$

Now we assemble all of the pieces to find,

$$\begin{aligned} &\delta_g S = \int_M d^4x\sqrt{-g}[\phi(G_{\mu\nu} + g_{\mu\nu}\Lambda) - \frac{\omega}{\phi}[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2] + ... \\ &...\frac{1}{\phi}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi + \frac{1}{2}g_{\mu\nu}V(\phi)] \to \\ &G_{\mu\nu} = \omega[\frac{\nabla_\mu\phi}{\phi}\frac{\nabla_\nu\phi}{\phi} - \frac{1}{2}g_{\mu\nu}(\frac{\partial\phi}{\phi})^2] - g_{\mu\nu}\frac{V(\phi)}{2\phi} + \frac{1}{\phi}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi + \frac{T^{(f)}_{\mu\nu}}{\phi} - g_{\mu\nu}\Lambda, \end{aligned} \tag{23}$$

where is the second line, we remember equation 2.1 and set everything inside of $\int_M d^4x\sqrt{-g}\delta g_{\mu\nu}[]$ to zero. This is our first equation of motion which we found from varying the action with respect to the metric. We can rewrite this equation of motion if we define the stress-energy tensor for the scalar field as,

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{\phi}[T^{(\phi)}_{\mu\nu} + T^{(f)}_{\mu\nu}] + \frac{1}{\phi}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi - g_{\mu\nu}\Lambda, \\ T^{(\phi)}_{\mu\nu} &= \frac{\omega}{\phi}[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2] - \frac{g_{\mu\nu}}{2}V(\phi), \end{aligned} \tag{24}$$

where $T^{(f)}_{\mu\nu}$ is to represent the upcoming stress-energy tensor for the fermion field which we shall derive in the next section. The reason we do not include the dynamic term on the far RHS of 2.17 is because it originally came from varying the Ricci tensor. For now, we are still missing one equation of motion, specifically varying the action with respect to ϕ. This is what we shall find next.

2.4 Variation with respect Scalar Field

We shall now take the action in equation 1.1 and vary with respect to the scalar ϕ. This resembles,

$$\delta_\phi S = \int_M d^4x\sqrt{-g}[(R - 2\Lambda)\delta\phi - \omega\delta(\frac{(\partial\phi)^2}{\phi}) - \delta V(\phi)] \tag{25}$$

The first term was simple since the only term present was ϕ, the second term takes a little work, but the third is simply, $\frac{dV}{d\phi}\delta\phi$. Now, that second term can be expanded as such,

$$\begin{aligned}\delta\left(\frac{(\partial\phi)^2}{\phi}\right) &= -\frac{\partial\phi}{\phi}^2 + \frac{g_{\mu\nu}}{\phi}\delta(\nabla_\mu\phi\nabla^\mu\phi),\\ &= -\frac{(\partial\phi)^2}{\phi} + 2\frac{\nabla_\mu\phi\nabla^\mu(\delta\phi)}{\phi}.\end{aligned} \tag{26}$$

And if we define some vector $A^\mu \equiv (\nabla^\mu\phi/\phi)\delta\phi$, then we can compute the last term as so,

$$\begin{aligned}&\nabla_\mu A^\mu = \frac{\Box\phi}{\phi}\delta\phi - \left(\frac{\partial\phi}{\phi}\right)^2\delta\phi + \frac{\nabla^\mu\phi}{\phi}\nabla_\mu(\delta\phi),\\ &\to \int_M d^4x\sqrt{-g}\delta\phi\left[R - 2\Lambda - \omega(\frac{\partial\phi}{\phi})^2 + 2\omega\frac{\Box\phi}{\phi} - \frac{\partial V}{\partial\phi}\right],\\ &R = 2\Lambda + \omega\left(\frac{\partial\phi}{\phi}\right)^2 - 2\omega\frac{\Box\phi}{\phi} + \frac{\partial V}{\partial\phi}.\end{aligned} \tag{27}$$

The second line is plugging in the results above it, and the final line is remembering, $\frac{\delta S}{\delta\phi} = 0$. This can be thought of as a Klein-Gordon equation of the Brans-Dicke field with a current. But, we can do better from here to find a relationship between matter, $T^{(f)}_{\mu\nu}$, and the dynamics of our scalar field. We can take the trace of our first equation of motion and solve for the ricci scalar R,

$$R = \omega\left(\frac{\partial\phi}{\phi}\right)^2 + \frac{2}{\phi}V(\phi) - \frac{T^{(f)}_{\mu\nu}}{\phi} + 3\frac{\Box\phi}{\phi} \tag{28}$$

And if we plug this into 2.19 and simplify,

$$T^{(f)} = 2V(\phi) + (3 + 2\omega)\Box\phi - 2\Lambda\phi - \phi\frac{\partial V}{\partial\phi}. \tag{29}$$

This is the last equation of motion we will find from the Brans-Dicke action. Next, we turn our attention towards our matter action being the fermion field. In the next section, we shall derive the equations of motion, as well as the stress-energy tensor. It will require first some formalism since different texts go about the process of defining spin connections differently; so we shall take it one step at a time to clarify everything.

2.5 Equations of Motion from Fermionic Field

We begin by first recalling that our action is of the form,

$$S = \int d^4x\sqrt{-g}\,\frac{i}{2}[(\bar{\Psi}\gamma^\mu\nabla_\mu\Psi \;\; - \Psi\gamma^\mu\nabla_\mu\bar{\Psi}) - m(\bar{\Psi}\Psi) - V(\bar{\Psi}\Psi) \tag{30}$$

From here, we shall define new quantities. The first being how we build a covariant version, or a curved spacetime version, from a flat spacetime theory. Since I stated the covariant version, let me first reference the flat spacetime version,

$$S = \int d^4x\,(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi) \tag{31}$$

The only difference is the $\frac{i}{2}$ which comes from expanding to show the adjoint spinor (also no potential). To scale up, first we realize the need to 'covariantize' the equation by promoting $\partial_\mu \to \nabla_\mu$. As in the same way one has found the covariant derivative of a spacetime manifold, this process is over a 'spinor manifold.' A much more in-depth analysis can be found in chapter nineteen of [7]. The fast paced version is what we find are similar to spacetime connections (christoffel symbols) which we call a spin connection. Except this was already mentioned in 1.4. What was not mentioned was that the spin connections were built out of tetrad or vielbein basis'. A relativistic example can be found in Appendix J of Carroll's General Relativity book [10]. They are denoted as e^{μ}_{a} *or* e^{a}_{μ} and formed by,

$$g_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} \eta_{ab} \, , \; \gamma^{\mu} = e^{\mu}_{a} \gamma^{a} \tag{32}$$

And on how to calculate tetrads, it can be easier to not directly solve it, but to think of how everything in the metric in curved spacetime, can be boiled down to 1. And in our specific case, since our metric is diagonal, what squared can equal 1 from the metric. The second equation in 2.25 is simply a specific way of how we shall be using the tetrad basis'. Since our old gamma matrices are from flat spacetime, we need to put them in a curved formalism; and it is surprisingly elegant. I will also point out that in this context, Greek symbols are 4d curved spacetime, English letters are to represent 4d flat spacetime, and only the i,j,k are used to represent the spatial components of the 4d curved spacetime.

Lets solve for the tetrads now since we shall recall upon them later on. It can be seen very straight forward remembering that we are using the Robertson-Walker metric, the tetrads are,

$$e^{a}_{\mu} = [1 \, , \frac{a(t)}{\sqrt{1-kr^2}} \, , a(t)r, a(t)rsin(\theta)] \equiv [e^{0}_{0}, e^{1}_{1}, e^{2}_{2}, e^{3}_{3}] \tag{33}$$

We can now begin to see, just as we can when viewing Christoffel symbols, what makes the spin connection, a connection,

$$\Omega_{\mu} = -\frac{1}{4} g_{\rho\sigma} [\Gamma^{\rho}_{\mu\delta} - e^{\rho}_{b} \partial_{\mu} e^{b}_{\delta}] \gamma^{\sigma} \gamma^{\delta} \tag{34}$$

Where $\Gamma^{\rho}_{\mu\delta}$ are the Christoffel symbols. From here, we can begin finding our equations of motion, and lastly our stress-energy tensor.

2.6 Variation with respect to $\psi/\bar{\psi}$

To find our first equation of motion, we can strict the variation principle to find our equations of motion and then take the complex conjugate of one to find the other. However, in regular field theory, we use the Euler-Lagrange equations to find the equations of motion. In this context for the fermion field, we can use the latter. This is because previously, our Brans-Dicke action (and the usual Hilbert action) is not written in terms of covariant derivatives of the metric [10].

With this in mind, the covariant version of the Euler-Lagrange equations are straight forward,

$$\nabla_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\nabla_{\mu}\psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{35}$$

So if we apply this to the lagrangian density of our fermion action, as well as take the complex conjugate, we find,

$$(i\gamma^{\mu}\nabla_{\mu} + m)\bar{\psi} + \frac{dV}{d\psi} = 0, \; (i\gamma^{\mu}\nabla_{\mu} - m)\psi - \frac{dV}{d\bar{\psi}} = 0 \tag{36}$$

From here, we shall now find our stress-energy tensor.

2.7 Stress-Energy tensor of fermion field

There are two methods for finding the stress-energy tensor. One is the strict method from general relativity, $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$. [10]. On the other hand, in QFT, a way of finding a symmetric stress energy tensor is,

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi)}\partial_\nu \psi - \eta_{\mu\nu}\mathcal{L} \tag{37}$$

This is the usual way you will find a symmetric stress-energy tensor in QFT in any textbook [6]. Or, for a quick and sufficient guide [11] which gives a nice presentation of stress-energy / energy-momentum tensors in classical gauge theories (since the covariant derivative in QFT is a gauge derivative). The article also takes the viewpoint of symmetry, making derivations very simple.

But what was not in the article, was the thought that since this is a tensorial equation, and the metric in use is $\eta_{\mu\nu}$. This can be thought of taking a tensor in general relativity, and being at a point P in a LIF (local inertial frame). This is what we did for finding the Palatini equation. We can then also use this equation, plug in our Dirac Lagrangian, find our stress tensor, and promote the partial derivatives to covariant derivatives (which is in the article previously mentioned [11], except for the derivative promotion part). Doing so gives the stress-energy tensor for the fermion field,

$$T_{\mu\nu} = \frac{i}{2}[\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi - \nabla_{(\mu}\bar{\psi}\,\gamma_{\nu)}\psi] - g_{\mu\nu}\mathcal{L}, \tag{38}$$

where we have used symmetric summation convention between gamma matrices and covariant derivatives.

If you follow the proper method, which can be found in [9], in where to solve the variational principle version, you state $g^{'}_{\mu\nu} \rightarrow = g_{\mu\nu} + h_{\mu\nu}$ where we make a variation of the metric and find the necessary quantities.

3 Friedmann and Acceleration Equations

In this section, we shall derive the equations of motion for a universe modeled by the FRW metric. This chapter will look very similar, and go about a similar process, that any introduction or advanced cosmology textbook introduces the Friedmann equation and acceleration equation. In a sense, these equations are the velocity and literal acceleration equations of the universe.

If one wants an introduction of how this process goes from a more simple example, then chapter twenty-six of Moore's GR book is a great reference [3].

3.1 Friedmann Equation for a BD Modeled Universe

To obtain the first Friedmann equation, we begin by taking equation 2.17 and find the $'00'$ signature, or setting $\mu = \nu = 0$ and find the expansion.

To begin, we split up 2.17 into separate derivations, this being first the Einstein Tensor, next the stress tensors, then the dynamics of the scalar field with our cosmological constant Λ. The Einstein tensor derivation can be found in many text as a reference, or if you want to double check their work, a great tool can be Moore's diagonal matrix calculation sheets [12] which resembles that of the timed multiplication tables of elementary school, except this time, for GR.

The Einstein tensor in the 00 signature takes the form,

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1)6(\frac{\ddot{a}}{a} + \frac{(\dot{a})^2}{a^2} + \frac{k}{a^2}) = 3H + \frac{3k}{a^2}, \tag{39}$$

where k is the curvature value taking on $-1, 0, or\, 1$ depending on an closed, flat, or open universe. Anything with a dot denotes a derivative with respect to time. We have also introduced the new symbol H which is the Hubble parameter in this case and is defined as $H \equiv \dot{a}/a$.

Next, we move onto the stress-energy tensors. For $T^{(\phi)}_{\mu\nu}$, it is relatively straightfoward that is since $\phi = \phi(t)$ only evolves with time, so any derivative other then time is 0. We also see that for both stress-tensors the '$00'$' signature gives an equation for pressure,

$$T^{(\phi)}_{00} = \frac{\omega}{\phi}\left[\dot{\phi}^2 + \frac{1}{2}(-\dot{\phi}^2)\right] + \frac{V(\phi)}{2} = \frac{\omega}{2}P\dot{\phi} + \frac{V(\phi)}{2} = \rho_\phi \tag{40}$$

We have again introduced a new quantity P that is defined similar to H, $P \equiv \dot{\phi}/\phi$. And for the stress-energy tensor of the fermion field, it is easier to first expand the equation, and see that everything cancels except for the potential and mass term coming from $g_{\mu\nu}\mathcal{L}$,

$$T^{(f)}_{00} = -m\bar{\psi}\psi - V(\Psi) = \rho_f \tag{41}$$

An interesting note is that the overall sign of the fermion stress tensor is negative. The last part is $-g_{00}\Lambda$ which is simply Λ. Now if we put all the pieces together, we find,

$$3H + \frac{3k}{a^2} = \frac{1}{\phi}(\rho_\phi + \rho_f) - 3HP + \Lambda \rightarrow H = \frac{1}{3\phi}(\rho_\phi + \rho_f) - HP + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{42}$$

This is our Friedmann equation that describes the dynamics of a universe filled with a Brans-Dicke scalar and fermions, along with a cosmological constant Λ.

As a side note, one might be puzzled as to why we have a cosmological constant with equations that are an attempt to explain the current issue of universe acceleration. It it because we want to find the dynamics when $\Lambda = 0$ and then include Λ when comparing to modern standards. Hopefully, without it, they match. If not, and the data only matches with Λ, then BD gravity in the presence of a fermion field is not the answer at least now in a precise manner, but in an analytical sense.

3.2 The Acceleration Equation

In a similar matter we can find the Friedmann equation or the acceleration equation by computing the '$11'$' or the 'rr'' signature of equation 2.17. Again, we break it apart in the exact same way.

For the Einstein tensor, we find,

$$\begin{aligned} G_{11} &= \left(-2\frac{\ddot{a}}{a} - H - \frac{k}{a^2}\right) g_{11}, \\ g_{11} &= \frac{a^2}{1 - kr^2}. \end{aligned} \tag{43}$$

For our stress-energy tensor,

$$\begin{aligned} &T^{(\phi)}_{11} = \frac{\omega}{\phi}\left[\partial_1\phi\partial_1\phi - \frac{1}{2}(\partial\phi)^2\right] - g_{11}\frac{v(\phi)}{2} \rightarrow \\ &\left(\frac{\omega}{2}\dot{\phi}P - \frac{V(\phi)}{2}\right) g_{11} = P_\phi g_{11}, \end{aligned} \tag{44}$$

for our scalar field stress-energy tensor. We have identified a pressure term, which we shall do also for the fermion field. This is,

$$T^{(f)}_{11} = \left(\frac{\bar{\psi}}{2}\frac{dV}{d\psi} + \frac{\psi}{2}\frac{dV}{d\bar{\psi}}\right) g_{11} = P_\psi g_{11}. \tag{45}$$

And for our scalar field dynamics (including Λ),

$$\frac{1}{\phi}(\nabla_1\nabla_1 - g_{11}) - g_{11}\Lambda = g_{11}(\frac{\ddot{\phi}}{\phi} + 3HP) - g_{11}\Lambda, \tag{46}$$

which, if we finally combine everything together, we find our acceleration equation,

$$-2\frac{\ddot{a}}{a} - H - \frac{k}{a^2} = \frac{1}{\phi}(P_\phi + P_f) + \frac{\ddot{\phi}}{\phi} + 3HP - \Lambda \tag{47}$$

This is not our final expression. Instead, if we plug in our expression for the Hubble parameter into the acceleration equation, then we arrive at our final expression.

$$\frac{\ddot{a}}{a} = -\frac{1}{6\phi}(3P - \rho) - HP + \frac{\Lambda}{3} - \frac{\ddot{\phi}}{\phi}. \tag{48}$$

We have combined the pressure and energy density terms into one single term. At a first glance we see that almost everything in our universe is slowing its expansion, except our cosmological constant. So if we do not include our cosmological constant, then our pressure or energy density term must be negative (and overcome one or the other).

We also have the term, $\ddot{\phi}/\phi$ is a direct reflection of our acceleration term $\ddot{a}$. We are still missing one more dynamical equation for our universe, which shall come from the thermodynamics.

3.3 Thermodynamics of our Universe

We still need to find our equation of motion from our Dirac equation and ϕ equation, but before that, we still have another equation, relating expansion with pressure and energy density. This comes from thermodynamics. The proper way, is from our conservation of energy equation in general relativity, $\nabla_\mu T^{\mu\nu} = 0$. Another less invasive (since we have two stress-energy tensors), can taken from an introduction to thermodynamics and statistical physics course. This is often called the Fluid equation.

The next section is taken from Ryden's book on cosmology [13]. So this will be slightly brief.

We first remember that the first law of thermodynamics is,

$$dQ = dE + PdV - \mu dN, \tag{49}$$

with dQ being the heat flow of the system. We also set the particle number dN to be constant, thus this term goes to zero. If we assume our particular universe has a boundary, on the edge, with no heat going in our out, then it can be considered adiabatic, in which case we take $dQ = 0$. We also will remind oneself that each derivative is actually with respect to time. So 3.11 becomes,

$$\dot{E} + P\dot{V} = 0. \tag{50}$$

And if one recalls the Newtonian version of the Friedmann equation, our volume is $V(t) = \frac{4\pi}{3}r_s^3 a(t)^3$. And if we remember the internal energy of a sphere being $E(t) = V(t)\rho(t)$. Lastly, if we take the time derivatives and plug in, we find,

$$\dot{\rho}_{tot} + 3H(\rho_{tot} + P_{tot}) = 0, \tag{51}$$

with ρ_{tot} and P_{tot} being the ϕ and fermion stress tensors added together. This is exactly the same as if we would have went the route of our conservation of energy, or our Bianchi identity for the stress-energy tensors.

3.4 Fermion equation of motion

For the fermion equation of motion, we start off with the Dirac equation we found,

$$(i\gamma^{\mu}\nabla_{\mu} - m)\psi - \frac{dV}{d\bar{\Psi}} = 0. \tag{52}$$

We also recall that the covariant derivative acting on the wave function ψ is not with a spacetime connection, but rather a spin connection,

$$\Omega_{\mu} = -\frac{1}{4} g_{\rho\sigma} \left[\Gamma^{\rho}_{\mu\delta} - e^{\rho}_{b} \partial_{\mu} e^{b}_{\delta} \right] \gamma^{\sigma}\gamma^{\delta}. \tag{53}$$

To simplify this derivation, we are going to assume that ψ only evolves with respect to time, and not r, θ, or ϕ (being the angle in this case). This greatly reduces the problem. Another simplification comes from finding the christoffel symbols. Again, Moore has a great resource for finding these easily [12]. At $\mu = 0$, the only christoffel symbols are,

$$\Gamma^{1}_{01} = \Gamma^{2}_{20} = \Gamma^{3}_{30} = \frac{\dot{a}}{a} = H. \tag{54}$$

This puts a constraint on the rest of the free indices as well. Annoyingly, all that work reduces to a spin connection of $\Omega_{\mu} = 0$. So all we have left is $\dot{\psi}$. Except, we still have to find $\mu = 1 = 2 = 3$ or r, θ, ϕ since it is an implied summation in 3.14. This is left as an exercise for the reader. A great outline of how this process is done (and using the symmetries of commutators) is [14]. We find our fermion equation of motion from the Dirac equation to be

$$\dot{\psi} - \frac{3}{2} H\psi + im\gamma_{0}\psi + i\gamma_{0}\frac{dV}{d\bar{\psi}} = 0. \tag{55}$$

3.5 ϕ Equation of Motion

Our last equation will come from the equation we derived back in chapter two with taking the variation of the action with respect to the Brans-Dicke scalar field. First, recall that out stress-energy tensor for our scalar is,

$$T^{(\phi)} = 2V(\phi) + (3 + 2\omega)\Box\phi - 2\Lambda\phi - \phi\frac{\partial V}{\partial \phi}. \tag{56}$$

We do not have a whole lot to do here other then find the trace of the fermion stress-energy tensor, and find the D'Almbertian operator acting on the scalar ϕ. We have already done the latter of these two, and will thus focus on the trace of the stress-energy tensor.

This process is rather simple, as we can first see from 2.32 that if we take the trace, the first portion becomes the exact same as the second half but times 4 (which comes from $g_{\mu\nu}g^{\mu\nu} = 4$). Then, again use the equations of motion (the Dirac equation) to see that we can sub in for the rest of the terms being,

$$\gamma^{\mu}\nabla_{\mu}\psi = -im\psi, \;\; \gamma^{\mu}\nabla_{\mu}\bar{\psi} = im\bar{\psi}. \tag{57}$$

This boils down to is $T^{(f)} = m\bar{\psi}\psi + 4V(\Psi)$. And from here, we can do some simplification, except to pull out the overall negative in the D'Almbertian operator. This leads us towards,

$$m_{f}\bar{\psi}\psi + 4V(\Psi) = -(3 + 2\omega)(\ddot{\phi} + 3H\dot{\phi}) - 2\Lambda\phi - \phi\frac{dV}{d\phi}. \tag{58}$$

This in itself is already an interesting equation. We have the dynamics (which will come solely from our potential) of fermions, equals that of the dynamics of our Brans-Dicke scalar field. The only glaring issue that sways us away from "fermion interactions equals scalar interaction" is the

Hubble constant H. Although this does not take away the relation stated above since the scalar field "velocity" is proportional to H, the "velocity" of the expansion of our universe.

All together in this section, we have found our equations of motion that will allow us to plot energy density as a function of time, as well as be able to model the acceleration of the universe with respect to time.

In measuring the energy density, the particular equations of interest will be,

$$\begin{aligned} &m_f\bar{\psi}\psi + 4V(\Psi) = -(3+2\omega)(\ddot{\phi}+3H\dot{\phi}) - 2\Lambda\phi - \phi\frac{dV}{d\phi}, \\ &\dot{\rho}_{tot} + 3H(\rho_{tot}+P_{tot}) = 0. \end{aligned} \tag{59}$$

And for studying the acceleration dynamics of our BD modeled universe, we shall be interested in,

$$\begin{aligned} &\frac{\ddot{a}}{a} = -\frac{1}{6\phi}(3P-\rho) - HP + \frac{\Lambda}{3} - \frac{\ddot{\phi}}{\phi}, \\ &\dot{\psi} - \frac{3}{2}H\psi + im\gamma_0\psi + i\gamma_0\frac{dV}{d\bar{\psi}} = 0. \end{aligned} \tag{60}$$

All together, they shall be used as a system of equations to solve for a(t), $\Psi(t)$, $\rho_{\phi,f}(t)$, and $\phi(t)$.

Next in the next section we shall be choosing a general potential to model the universe. We shall be using a typical ϕ potential from ϕ^4 theory, and then a general potential taken from Rakhi's paper [15], which creates a potential out Lorentz invariant quantities created from ψ functions. The main difference though is how we shall compare results; which will be discussed at the beginning of section five.

4 Choosing our Potentials

In this section, we shall pick and evaluate our potentials for $V(\phi)$ and for $V(\Psi)$. This will be our last step to finally begin to evaluate and interpret the results.

We shall begin with the easier of the two, $V(\phi)$.

4.1 ϕ Potential

In our case, we are exploring the option of whether or not we can explain either early universe expansion, or late universe expansion from introducing a scalar field proportional to Newton's gravitational constant $G \propto \frac{1}{\phi}$ without including extra terms or extra explanations such as inflation. I will note we do have Λ in our derivations, but this is for comparison to see if it affects or fixes our predictions.

But since we are defining a scalar as such, we believe it would be poor to begin to introduce potentials that may have no physical meaning towards the way we defined ϕ. Hence, we pick our potential to be the very common case,

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{61}$$

4.2 General Fermion Potential

The general fermion potential we shall use, as referenced before, is found in [15],

$$V(\psi\bar{\psi}) = \lambda[\beta_1(\bar{\psi}\psi)^2 + \beta_2(i(\bar{\psi}\gamma^5\psi))^2]^n \tag{62}$$

Where $V(\bar{\psi}\psi)$ is constructed out of Lorentz invariant quantities (tensor bilinears). We could add more quantities, which would require more parameters such as $\bar{\psi}\psi$. This would start to become complicated since it's parity and temporal symmetry start to take different values depending on $(-1)^\mu$. This potential also has a specific name for when $n = 1$ being the Nambu-Jona-Lasinio potential.

5 Analysis and Results

In this final section, we will look at various scenarios and read off interactions. The possible scenarios are going to be when the mass terms in ϕ and ψ both zero, either or 0 or positive, both positive, what happens to our potential and overall evolution, and for when $\Lambda = 0$ or positive. A standing point is that the dependence of curvature k in the acceleration equation disappears, but remains the exact same in the first Friedmann equation,

$$H = \frac{1}{3\phi}(\rho_\phi + \rho_f) - HP + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{63}$$

as it would be if we disregarded the interactions of the BD scalar field. This is comforting since it means that curvature is not dependent on a specific field, yet still is a necessary measured value, not predicted. We will also be looking at when β_1 and β_2 take values of $0, 1$ and each variation. However, it appears that the new term, $-HP$, shows that the BD scalar is tied to the dynamics of the scale factor. This means that if we were to model a closed or open universe, the closed universe would close faster, and the open would expand faster towards infinity.

Consider for instance first how the energy density and pressure of the fermions will become in equation 41 and 45 if we plug in the potential from equation 62,

$$\begin{aligned} -\rho_f &= m(\bar{\psi}\psi) + \lambda[\beta_1(\bar{\psi}\psi)^2 + \beta_2(i(\bar{\psi}\gamma^5\psi))^2]^n, \\ p_f &= (2n-1)\left[m(\bar{\psi}\psi) + \lambda[\beta_1(\bar{\psi}\psi)^2 + \beta_2(i(\bar{\psi}\gamma^5\psi))^2]^n\right], \\ &= (2n-1)\left[\rho_f - m(\bar{\psi}\psi)\right]. \end{aligned} \tag{64}$$

We can classify the fermions based upon the parameter n. Indeed, when this is done, we find that for $n \geq 1/2$, the fermions will represent a matter field with a positive pressure for when $n \geq 1/2$; since the the first term in the pressure of equation 64 will dominate the second term. But if we are equal to one, then we find a pressureless fluid since the terms cancel. Finally, if $n \leq 1/2$, then we have a negative pressure field which will either represent inflation or dark energy[11].

Now consider if the fermions are massless. We still have the same scenarios as stated directly above, but the pressure simplifies to a simple relationship to the density,

$$p_f = (2n-1)\rho_f, \tag{65}$$

which from our conservation equation 59 giving $\rho_f \sim 1/a^{6n}$. In this particular case, when $n > 1/2$, the fermionic field will not differ from the matter field containing the scalar, and this is true also for when $n < 1/2$ (when the fermions behave as bosons). Thus, the only interesting case is when we have a mass term for the fermions, and when $n < 1/2$.

To be able to analyze this case (and the ones we skipped in the paragraph directly above), we need to write out the spinor components of the Dirac equation number 60 which if, $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T$

will be written out as,

$$\frac{d}{dt}\begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{bmatrix} - \frac{3H}{2}\begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{bmatrix} + im\begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{bmatrix} ...$$

$$... - 2i\left(\psi_0^\dagger\psi_0 + \psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2 - \psi_3^\dagger\psi_3\right)\frac{dV}{d\left((\psi\bar{\psi})^2\right)} - 2i\left(\psi_2^\dagger\psi_0 + \psi_3^\dagger\psi_1 - \psi_0^\dagger\psi_2 - \psi_1^\dagger\psi_3\right)\frac{dV}{d\left((\psi\gamma^5\bar{\psi})^2\right)} = 0. \tag{66}$$

Now, in order to obtain a numerical value, we can define specific values in which we wish to analyze. From the previous discussion on the value of n and the mass m we want a value less then $n < 1/2$ so we can choose $n = 1/4$. As for our mass term, we want something greater then 0, so we choose $m_f = 0.01$. Recall, we still have two different values within the potential, $\beta_{(1,2)}$ which, for our initial case will be $\beta_1 = \beta_2 = 1$ (then we will include both invariant terms).

For the initial conditions of our universe, we can adjust our clocks and choose $t = 0$[13]. This will then give the following initial conditions for the scalar factor, fermions, and scalar-energy density,

$$a(0) = 1,\ \psi_0(0) = 0.1i,\ \psi_1(0) = 1,\ \psi_2(0) = 0.3,\ \psi_3(0) = i,\ \rho_m(0) = 0.05. \tag{67}$$

Under numerical analysis, the energy density peaks to 0.014 at initial time, while the matter-scalar density peaks at 0.05 at initial time. Both densities decrease exponentially but never reach zero. This implies an initial acceleration at early times. This entire analysis is without including the term $\lambda = 0$.

Now, our another analysis would be changing the values of $\beta_{(1,2)}$. However, doing this brings no new dynamics that would give physically interesting values. If both $\beta_1 = \beta_2 = 0$, then we must include a the parameter $\Lambda \neq$ in order to recover the same dynamics described above for dark energy, and let the scalar potential be non-zero (which was zero previously). We also return to the typical model found in inflationary dynamics, only at the typical value[13] and [5].

It should be noted that these dynamics are of an effective field theory model with respect to high energies at early times in the unviverse. However, effective field theories arising from quantum fields in curved spacetime are subject only to energy limits that of the Planck scale. But, in this particular case, if we are viewing the dynamics of the fermionic potential to be a substitute for inflation, then the energies would be that of the Planck scale during this short time as described by typical inflation models (but of course a reheating phase is necessary, a part in which we did not go into).

6 Conclusion

In this paper, we analyzed Brans-Dicke gravity, a substitute to Einstein's theory of gravitation that upholds Mach's principle more suitably then the latter. With this theory, we added on a stress-energy term in the form of a fermionic field in curved spacetime. We then derived the equations of motion using the variational principle in explicit manner, and gave multiple ways to do so in particular steps. Next, we found the Friedmann equations that govern this particular model of the universe. In order to analyze these equations though, we had to define a potential for both in which the fermionic potential garnered more attention. In the analysis, we found that depending upon the exponetial value n within the fermionic value (given that both $\beta_1 = \beta_2 = 1$), we find either inflationary dynamics or dark energy dynamics at late times.

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