Covariant Loop Quantum Gravity:
Partial Solutions to the textbook by Rovelli & Vidotto

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Abstract

These are partial solutions towards the textbook, Covariant Loop Quantum Gravity by Rovelli and Francesca. Most of them are completed, but some chapters may be missing a few problems or parts of problems (mainly due to time constraints). Before using these solutions, of course firs try the problems on your own, and then check with mine, or even correct mine! As of August 14^{th}, 2021, these solutions are currently being worked on with the goal of one chapter done per week. These solutions are uploaded on my website: alexcassem.net on the blog tab.

Contents

1 Chapter 1: Spacetime as a quantum object

Exercise 1.1: Show that the definitions above are all equivalent.

The definitions in question are:

1. "the vectors $\vec{L_a}$ are (outgoing) normals to the faces of the tetrahedron and their norm is equal to the area of the face."

   What this means is that, $\|L_a\| = \|\frac{1}{2}(\ell_1 \times \ell_2)\|$.

2. "The matrix of the components L_a^i for $a = 1, 2, 3$ (notice that only 3 edges are involved) is $L^T = -\frac{1}{2}(det(M))M^{-1}$."

This initially did not seem straight forward, but with a little bit of geometrical interpretation (and hand-waviness) we can show the two are the same.

To show they are the same, I will construct from the ground up on how to define L_a, from which the definitions above will fall out. There are two pieces we first need, M is the matrix formed by the components *3 vectors* stemming from the same point, $M = (\ell_1, \ell_2, \ell_3)$. This is a volume, which can be computed from the parallelogram formula, $(\ell_1 \times \ell_2) \cdot \ell_3$, which is also the determinant, $det(M)$. If we take $\frac{1}{2}$ then we get the volume of a triangle, $V(\tau) = \frac{1}{2}det(M)$.

(Now for the hand-waviness), recall what the tranpose does to a vector, it (about) rotates while also expanding or contracting the vector to the adjacent quadrant. And if multiplied by the matrix which determines the components of the area for a side of the tetrahedron, then we can safely say, $L^T M \approx \vec{L_a}$. And as you can see, if we take the $p = 2$ norm (or just the regular norm we all know), we find the first definition.

We can also say that $V(\tau) \sim L^T M$, which then gives us the second definition.

Exercise 1.2: Prove these relations. *Hint: choose a tetrahedron determined by a triple of orthonormal edges, and then argue that the results are general because they are invariant under linear transformations.* Derive the $\frac{2}{9}$ factor.

There are six definitions in question, some are trivial, but some are fun exercises, so I will do three here:

1. *Verify that the outgoing normals of the surface satisfy the closure relation:* $\vec{C} := \sum_{a=1}^{4} \vec{L_a} = 0$.

   This can actually be shown or thought about if you let the area of each normal, $\vec{L_a}$ be the same, and then cut out 4 pieces of paper! The only way they can close then is if they form into a tetrahedron. But $\vec{C} = 0$ is only valid for the entire summation of each normal.

   The proper way to do this is to define $\vec{L_a}$ in terms of arbitrary numbers (a, b, c) and then use the definition given previously to sum them up: $\vec{L_a} = \frac{1}{2}(\ell_1 \times \ell_2)$ (each component will cancel out).

2. *The area A_a of the face a is $\|\vec{L_a}\|$.*

   We actually already showed this previously given the area of the face is related to the cross product: $\|L_a\| = \|\frac{1}{2}(\ell_1 \times \ell_2)\|$. To double check, you can plug in arbitrary vectors.

3. *The volume V is determined by the (properly oriented) triple product of any three faces*:

$$V^2 = \frac{2}{9}\left(\vec{L_1} \times \vec{L_2}\right) \cdot \vec{L_3} = \frac{2}{9}\epsilon_{ijk} L_1^i L_2^j L_3^k = \frac{2}{9} det(L).$$

   We actually have already (sort of) did this one before, except previously we had $\frac{1}{2}det(M)$ (by a numerical factor here and there).

So, the way the volume is computed is familiar. But the $\frac{2}{9}$ is tricky, which can be put in by hand since the normal volume is $V = \frac{1}{6}det(M)$, where we now make the fact of the $\frac{1}{2}$ evident by pulling it out of the determinant. And if V^2, then we get a factor of $\frac{1}{36}$, but we are double and triple counting for first: the edges that make up an area (the 2) and the three vectors from a single point that make up the original M matrix (the 3), giving us $\frac{2}{9}$.

Exercise 1.3: Show that $\mathcal{H}_j$ has dimension $2j + 1$.

The simplest way to show this is to recall just before the question the relationship between an n-index spinor, $z^{(A_1 \ldots A_n)}$, and the spin number $j = n/2$. If we have an $n = 2$ index spinor, the space should be 2-dimensional, which means the Hilbert space representing the spin state should also be, $\mathcal{H}_{2(2/2)+1} \to \mathcal{H}_2$. But, the space is 2-dimensional since there are two spin values (not based on dimensionality of the spinor). We can assume this trend continues.

Exercise 1.4: Show that (1.30) implies (1.29).

The equations in question are,

$$2j_1 = b + c, \;\; 2j_2 = c + a, \;\; 2j_3 = a + b,$$

and

$$j_1 + j_2 + j_3 \in \mathbb{N} \;\; \& \;\; |j_1 - j_2| \leq j_3 \leq (j_1 + j_2) \, .$$

This is a straightforward logic exercise since if you plug in the j_i relations into the above, you get, $a + b + c \in \mathbb{N}$, and same with the second one:

$$|b - a| \leq a + b \leq (2c + a + b) \, ,$$

which simply exists as a constraint towards the statement: there exists three non-negative integers that obey the first relationships. Thus, the constraints regarding the 3 integers just so happen to be the Clebsch-Gordon coefficients.

There is a second part to this problem to draw/show the pictorial description of each state j_i could be in (page 25), but for now, I will leave this as is.

Exercise 1.5: Show that the Pauli matrices obey:

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \;\; \& \;\; \epsilon\sigma_i\epsilon = \sigma_i^T = \sigma_i^* .$$

The first one is from the commutator and anticommutator relationships between pauli matrices, $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ & $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. Now, add the commutator relationships,

$$\begin{aligned} [\sigma_i, \sigma_j] + \{\sigma_i, \sigma_j\} &= 2i\epsilon_{ijk}\sigma_k + 2\delta_{ij}, \\ \sigma_i\sigma_j - \sigma_j\sigma_i + \sigma_i\sigma_j + \sigma_j\sigma_i &= 2\sigma_i\sigma_j, \\ \sigma_i\sigma_j &= \delta_{ij} + i\epsilon_{ijk}\sigma_k. \end{aligned} \tag{1}$$

The final relationship is of careful construction. First, recall the the spinor matrices:

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{AB} \tag{2}$$

Now, lets try out the σ_y matrix to see if we get the tranpose and the complex conjugate back after "pressing" two spinor indexes next to it,

$$\epsilon^{AB}\sigma_{yB}^{C}\epsilon_{CA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y. \tag{3}$$

The other Pauli matrices follow suit. This is just showing that the Pauli matrices are invariant under the product of spinors.

Exercise 1.6: Show that,

$$e^{i\alpha\hat{n}\cdot\hat{\sigma}} = cos(\alpha) + i\hat{n}\cdot\hat{\sigma}sin(\alpha).$$

Every time you may be given an Euler identity, rest can be assured, the extra factor (as long as it is a scalar) can be multiplied on the sign function. Although, lets be careful here, set $\hat{\sigma}\cdot\hat{n} = M$ with M being the matrix values of the product. Now, lets use the expansion of the matrix exponential,

$$\begin{aligned} e^{iM\alpha} &= \sum_{j=0}^{\infty}\frac{(iM\alpha)^j}{j!} = \sum_{j_{even}=0}^{\infty}\frac{(i\alpha)^j}{j!}M^j + \sum_{j_{odd}=0}^{\infty}\frac{(i\alpha)^j}{j!}M^j, \\ &= \sum_{j_{even}=0}^{\infty}\frac{(i\alpha)^j}{j!}I + \sum_{j_{odd}=0}^{\infty}\frac{(i\alpha)^j}{j!}M, \\ &= cos(\alpha)I + iMsin(\alpha), \\ &= cos(\alpha)I + i\hat{\sigma}\cdot\hat{n}sin(\alpha). \end{aligned} \tag{4}$$

In the first line, we separated the even and odd terms, on the second line we recognized that the square of the matrix is 1 (or the identity matrix), and then finally substituted the Taylor series expressions for their cosine and sine counterparts.

Exercise 1.7: Verify that they satisfy the following relation which defines the algebra SU(2),

$$[\tau_i, \tau_j] = \epsilon_{ij}^{k}\tau_k$$

This is pretty straightforward using the commutator relationship for Pauli matrices defined in exercise 1.5 since the generators are created out of: $\tau_k = -\frac{i}{2}\sigma_k$,

$$\begin{aligned} [\tau_i, \tau_j] = \tau_i\tau_j - \tau_j\tau_i &= \left(\frac{-i}{2}\right)^2(\sigma_i\sigma_j - \sigma_j\sigma_i) \\ &= \frac{1}{4}2i\epsilon_{ijk}\sigma_k \\ &= \frac{1}{2}2\epsilon_{ijk}\left(\frac{-i\sigma_k}{2}\right) \\ &= \epsilon_{ijk}\tau_k. \end{aligned} \tag{5}$$

The first line is just expanding the commutator and plugging in the relationship between the Pauli matrices and the generators of SU(2). The second line plugs in the Pauli commutator relationship, the third line is pulling out the two, and finally plugging in the definition of the generator again.

2 Chapter 2: Physics without time

Exercise 2.1: Show that the Hamilton function of an harmonic oscillator is,

$$S(q, t, q', t') = m\omega\frac{(q^2 + q'^2)\,cos(\omega(t'-t)) - 2qq'}{2sin(\omega(t'-t))} \tag{6}$$

I am going to first clean the Hamilton function a bit:

$$\begin{aligned} S(q, t, q', t') &= \frac{m\omega}{2}\left(q^2 + q'^2\right)cot(\omega(t'-t)) - m\omega qq'csc(\omega(t'-t)) \\ &= \frac{m\omega}{2}\left(q^2 + q'^2\right)cot(\omega\tilde{t}) - m\omega qq'csc(\omega\tilde{t}). \end{aligned} \tag{7}$$

Where we have used the trig. identities for cosecant and cotangent, and defined $\tilde{t} = t' - t$. We can also rearrange the general formula being equation 2.4 by differentiating with respect $d/d\tilde{t}$ to give,

$$d\mathcal{S}_f/d\tilde{t} = \mathcal{L}\left(q_{qt,q't'}(\tilde{t}), \dot{q}_{qt,q't'}(\tilde{t})\right) \rightarrow \frac{\partial \mathcal{S}_f}{\partial \tilde{t}} + \mathcal{L} = 0, \tag{8}$$

where the sign convention is also present in the lagrangian on the harmonic oscillator part. This allows us to take the derivative of the Hamilton function given to us in equation 2.8 and compare with the lagrangian density which, for a harmonic oscillator, should be,

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{9}$$

Now, we just have to take the derivatives of the given Hamilton function being,

$$\begin{aligned} \frac{\partial \mathcal{S}_f}{\partial \tilde{t}} &= -\frac{m\omega}{2}(q^2 + q'^2)csc^2(\omega\tilde{t}) + m\omega^2 qq'csc(\omega\tilde{t})cot(\omega\tilde{t}) \\ \frac{\partial \mathcal{S}_f}{\partial q} &= m\omega q cot(\omega\tilde{t}) - m\omega q' csc(\omega\tilde{t}), \end{aligned} \tag{10}$$

which we can use to plug the position derivative into out Lagrangian (which is looking more and more like the Hamiltonian) to see if it matches the time derivative:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}_f}{\partial q}\right)^2 + m^2\omega^2 q^2\right] \\ &= \frac{1}{2m}\left[m^2\omega^2(q^2 + q'^2)csc^2(\omega\tilde{t}) - 2m^2\omega^2 qq'csc(\omega\tilde{t}cot(\omega\tilde{t}))\right] \\ &= \frac{m\omega^2}{2}(q^2 + q'^2)csc^2(\omega\tilde{t})) - m\omega^2 qq'csc(\omega\tilde{t}cot(\omega\tilde{t})) \\ &= -\frac{\partial \mathcal{S}_f}{\partial \tilde{t}}, \end{aligned} \tag{11}$$

which cancels out in equation 8, thus satisfying equation 2.4 in the text.

There is another way by using the original formalism of equation 2.4 and solving for the Hamilton function directly, but you have to utilize the solutions to the *Hamiltonian*, namely,

$$q(\tilde{t}) = Asin(\omega\tilde{t} + \phi) \;\; \& \;\; p(\tilde{t}) = m\omega Acos(\omega\tilde{t} + \phi), \tag{12}$$

to find the matching Hamilton function above.

Exercise 2.2: *Try!* Or in other words, show that for the action of equation 2.22 to be independent of $\delta\dot{q}(0)$, you must impose the correct boundary terms in 2.24.

To show this, we simple take the variation of the action (we will do both),

$$S[q] = \int dt\, \frac{1}{2}mq\ddot{q} \;\; \& \;\; S[q,p] = \int dt\, \left(p\dot{q} - \frac{1}{2m}p^2\right) \tag{13}$$

and then see what boundary terms are required to keep the actions independent of $\delta\dot{q}(0)$ (which hopefully be that of, $S_{boundary} = mq\dot{q}|_{boundary} = pq|_{boundary}$, equation 2.24).

We could evaluate this integral two different ways; first, we simply evaluate the integrals, and see what boundary conditions are necessary to recover equation 2.5. Or, we could take the variation of the action and see what boundary conditions are required to have $\delta S/\delta\dot{q} = 0$. We shall use the first method (it is more direct). The first integral becomes,

$$S[q] = \int dt\, \frac{1}{2}mq\ddot{q} = mq\dot{q}|_{boundary} - \int dt\, \frac{m}{2}\dot{q}^2, \tag{14}$$

thus, we would need to require the term $mq\dot{q}|_{boundary}$ to go to zero to recover equation 2.5 (we performed integration by parts). For the second integral, the situation is similar,

$$s[q,p] = \int dt \left(p\dot{q} - \frac{1}{2m}p^2 \right) = pq|_{boundary} - \int dt \, \frac{m}{2}\dot{q}^2, \tag{15}$$

(we have evaluated the integral directly), thus we would need to require $pq|_{boundary}$ to go to zero to recover equation 2.5. You *could* take the variation and utilize the Euler-Lagrange method, but it seems overkill for this.

As a final note before moving onto chapter three (since there are only two problems for this chapter), the main focus of this chapter is to show how the Hamiltonian appears under discrete circumstances, and then the discrete Hamiltonian, called the *Hamilton* function, shows how to find the precise location of where you "ended up." This can already be viewed as a large step towards quantum mechanics, since we need discrete energies, and well if we can put Relativity in terms of the Hamiltonian, this looks like a nice step towards that goal.

3 Chapter 3: Gravity

These *exercises* are a the end of the chapter under section 3.6.2. There is only one exercise throughout the chapter, so we may have to find some from Rovelli's previous textbook (if we have the time)...

Exercise 3.1: Derive these equations of motion from the action (this relates to page 75, discussing equation 3.98).

This problem is asking us to find the equations of motion from the following action in D=3 Euclidean space,

$$S[e,\omega] = \frac{1}{16\pi G} \int \epsilon_{ijk} \, e^i \wedge F^{jk}[\omega], \tag{16}$$

where e^i is our tetrad, F^{jk} is our curvature two-form given in 3-dimensions as, $F^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$, and ω (in this context) is our *general* connection coefficient. For this problem, we will have two equations of motion, one showing that the torsion vanishes, and the other showing the curvature vanishes (as it should). Treating the tetrad and connection as independent will help the process.

We can vary each term directly, then simplify with the equations given on page 78 to give both the vanishing curvature and torsion tensor that we desire,

$$\begin{aligned} S[e,\omega] &= \int \epsilon_{ijk} \, e^i \wedge F^{jk}[\omega], \\ \delta S[e,\omega] &= \int Tr\left[\delta e \wedge F\right] + \int Tr\left[e \wedge \delta F\right] \\ &= \int Tr\left[\delta e \wedge F\right] + Tr\left[e \wedge D\delta\omega\right] \\ &= \int Tr\left[\delta e \wedge F\right] + Tr\left[T \wedge \delta\omega\right]. \end{aligned} \tag{17}$$

From the first variation, this implies $\epsilon_{ij}F^{ij} = 0$ which is vanishing of curvature, and the second part implies, $\epsilon_i T^i = 0$ implying that the torsion also disappears **if** we treat both the tetrad e^I and the connection ω^{ij} as independent variables. This is called the *1.5-order formulation* at the bottom of page 78.

Exercise 3.2: Derive the Dirac equation from the Dirac action on a curved spacetime.

The action the text refers to is equation 3.20 being,

$$\mathcal{S}_f[\psi, e] = \int \bar{\psi}\gamma^I D\psi \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL} = \int \bar{\psi} D\psi \wedge (e \wedge e \wedge e)^* \tag{18}$$

where the star, *, denotes the Hodge Product. Now, there are a few ways to do this, the first it to directly evaluate the integral and vary with respect to ψ, or we re-formulate the action that is currently in terms of tetrads and put it in terms of the metric $g_{\mu\nu}$. I will point out a glaring issue though, in my opinion, which is the absence of mass term. Presumably, we could simply add a term such as $m\psi\bar{\psi}$ onto the end, but without an extensive amount of knowledge on tetrads, I would feel uneasy about this.

But, how about we show how to recover the Dirac action under the metric, and then take the variation from this (it will turn out to be easier). To do this, recall how we construct gamma matrices in curved spacetime, $\Gamma^\mu = e^\mu_I \gamma^I$. I prefer Γ^μ instead of the usual γ^μ since it keeps things straight on whether we are in flat or curved spacetime, but I will keep the convention of the book and use γ^μ. With this, we will follow the same path as in the last problem of this section by expanding the tetrads via their definition, find the volume element, transform it from curved spacetime in terms of tetrads, and simplify:

$$\begin{aligned} S[\psi, e] &= \int \bar{\psi}\gamma^I D\psi \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL} \\ &= \int \bar{\psi}\gamma^\mu_I e^I_\mu (D_\mu \psi) dx^\mu \wedge e^J_\alpha dx^\alpha \wedge e^K_\beta dx^\beta \wedge e^L_\delta dx^\delta \epsilon_{IJKL} \\ &= \int \bar{\psi}\gamma^\mu_I (D_\mu \psi) e^I_\mu e^J_\alpha e^K_\beta e^L_\delta \epsilon_{IJKL} (dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\delta) \\ &= \int \bar{\psi}\gamma^\mu_I (D_\mu \psi) e^I_\mu e^J_\alpha e^K_\beta e^L_\delta \epsilon_{IJKL} \epsilon^{\mu\alpha\beta\delta} d^4x \\ &= \int \bar{\psi}\gamma^\mu_I (D_\mu \psi) \sqrt{-g} d^4x. \end{aligned} \tag{19}$$

This is the exact same process we outline below, but skipped a few since they are repeated for when deriving the Einstein-Hilbert action, the only difference is when finding the tensor-weight in terms of the metric which is a combination of the tetrad and transformation of coordinate frames using the totally-antisymmetric tensor. However, for us to full recover Dirac's equation on page 61 (equation 3.10), we have to put in by hand a mass term. When we do this, we shall then take the variation with respect to ψ, from which we shall recover the equation,

$$\begin{aligned} \delta S &= \delta \int d^4x \sqrt{-g}\, \bar{\psi}\gamma^\mu (D_\mu \psi) - \delta \int d^4x \sqrt{-g} m (\bar{\psi}\psi) \\ &= \int \left[d^4x \sqrt{-g} D_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} \right] \delta\psi, \end{aligned} \tag{20}$$

from which we shall get,

$$(\gamma^\mu D_\mu + m)\, \bar{\psi} = 0, \tag{21}$$

The once concern I do have is that when switching between the tetrad frame and the metric frame, we did not recover the i that is usually in front of the gamma matrix. Perhaps it was an error on my part (more then likely), or something we need to put in by hand.

Exercise 3.3: Develop the formalism discussed in Section 2.5.2 for the tetrad-connection formulation of general relativity.

Maybe later on....

Exercise bonus: Rewrite the Einstein-Hilbert action in the tetrad formulation (this is a suggestion on page 61, so why not).

So, I think the best route is to first show the relationship between the Ricci scalar and the curvature 2-form, and dwindle this down to only tetrads, and then start from equation 3.17 on page 61 and put it in an explicit form in terms of tetrads and curved spacetime indices.

First, recall the Einstein-Hilbert action,

$$S[g] = \frac{1}{2}\int d^4x\sqrt{-g}R = \int d^4x\sqrt{-g}R_{\mu\nu}g^{\mu\nu}. \tag{22}$$

Now, we can use the identity on page 61, equation 3.15 and rewrite the Riemann tensor as the Ricci tensor by contracting via tetrads,

$$R^{\mu}_{\nu\rho\sigma} = e^{\mu}_{I}e^{J}_{\nu}F^{I}_{J\rho\sigma} \; or \; F^{IJ}_{\rho\sigma} = e^{I}_{\mu}e^{J}_{\nu}R^{\mu\nu}_{\rho\sigma}, \tag{23}$$

if we drop the differential on equation 3.16. Now, we can expand equation 22 more, and also plug in $e = det(e) = \sqrt{-g}$,

$$S = \int d^4x\sqrt{-g}R_{\mu\nu}g^{\mu\nu} = ed^4xe^{\mu}_{I}e^{\nu I}R_{\mu\nu\rho\sigma}e^{\rho}_{J}e^{\sigma J} = \int ed^4xe^{\mu}_{I}e^{\rho}_{J}F^{IJ}_{\mu\rho}. \tag{24}$$

Now we could make a couple assumptions about the geometry and jump from equation 24 to to equation 3.17 in the text, but it will be better to understand what is going on in 3.17, so lets first expand, then contract, and see if we recover equation 24.

$$\begin{aligned}
S_{EH}[e] &= \frac{1}{2}\int \epsilon_{IJKL}e^{I}\wedge e^{J}\wedge F^{KL}(\omega(e)) \\
&= \frac{1}{2}\int \epsilon_{IJKL}e^{I}_{\alpha}dx^{\alpha}\wedge e^{J}_{\beta}dx^{\beta}\wedge \frac{1}{2}(F^{KL}_{\delta\gamma}\wedge dx^{\delta}\wedge dx^{\gamma}) \\
&= \frac{1}{4}\int \epsilon_{IJKL}e^{I}_{\alpha}e^{J}_{\beta}F^{KL}_{\delta\gamma}(dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\delta}\wedge dx^{\gamma}) \\
&= \frac{1}{4}\int d^4x\epsilon^{\alpha\beta\delta\gamma}\epsilon_{IJKL}e^{I}_{\alpha}e^{J}_{\beta}F^{KL}_{\delta\gamma} \\
&= \frac{1}{4}\int d^4xe\,\epsilon_{IJKL}\epsilon^{I'J'K'L'}e^{\alpha}_{I'}e^{\beta}_{J'}e^{\gamma}_{K'}e^{\delta}_{L'}e^{I}_{\alpha}e^{J}_{\beta}F^{KL}_{\gamma\delta} \\
&= \frac{1}{4}\int d^4xe\,\epsilon_{IJKL}\epsilon^{I'J'K'L'}\delta^{I}_{I'}\delta^{J}_{J'}e^{\gamma}_{K'}e^{\delta}_{L'}F^{KL}_{\gamma\delta} \\
&= \frac{1}{4}\int d^4xe\,\epsilon_{IJKL}\epsilon^{IJK'L'}e^{\gamma}_{K'}e^{\delta}_{L'}F^{KL}_{\gamma\delta} \\
&= \frac{1}{4}\int d^4xe\,2(\delta^{L}_{K}\delta^{L'}_{K'} - \delta^{L'}_{K}\delta^{L}_{K'})e^{\gamma}_{K'}e^{\delta}_{L'}F^{KL}_{\gamma\delta} \\
&= \int d^4xe\,(\delta^{L}_{k}\delta^{L'}_{K'})e^{\gamma}_{K'}e^{\delta}_{L'}F^{KL}_{\gamma\delta} \\
&= \int d^4xe\,e^{\gamma}_{K}e^{\delta}_{L}F^{KL}_{\gamma\delta}.
\end{aligned} \tag{25}$$

Ok, the first line is just equation 3.17, the next line is plugging in the definition for tetrads, $e^I = e^I_{\mu}dx^{\mu}$ and the curvature two-form. Line three is taking out all the wedge product, which we recognize to be the volume element of a manifold in four dimensions and simplify to $\epsilon^{\alpha\beta\gamma\delta}d^4x$ in line four. Line five is us recognizing that we previously plugged in the totally-antisymmetric tensor in 4-dimensions for curved spacetime constructed under the metric, so we must transform it to the tetrad version. Line six is finding delta function, line seven is simplyfying via the delta functions, line eight is using a kronecker-delta identity to further simplify down to the very last line, which does agree with what we had directly above it for putting the Riemann tensor in terms of tetrads.

4 Chapter 4: Classical discretization

Exercise 4.1: Compute the equations of motion of the discretization of the covariant formulation of a Newtonian system. Is the energy conserved?

Exercise 4.2: Consider a tetrahedron. Place a point P in its interior and connect P with the four vertices of the tetrahedron. Let the resulting ten segments determine a Regge space. Carefully describe the corresponding triangulation Δ and its dual Δ^*, listing all vertices, edges, and faces, distinguishing those that are similar or are of different kinds. Write the boundary graph. What is the relation between the boundary graph and the original tetrahedron? Consider then the group elements associated with all the edges of Δ^*. Find a good notation to label these. How many conditions must these satisfy in order for the connection to be flat? (Careful: there is a bubble...)

Exercise 4.3: Consider two spacetime regions connected by a common portion of their boundary. Let both of them be triangulated, matching at the boundary. Describe how the dual triangulations match at the boundary. What happens to nodes and links? Build a concrete example giving all the vertices, edges, faces, nodes, and links explicitly.

Exercise 4.4: Show that the various definitions of the holonomy given above are equivalent. Derive (4.45), (4.46), and (4.47) from the formal definition of the holonomy.

Exercise 4.5: What is the relation between the Regge curvature defined by the deficit angle δ_h and the curvature defined by U_f? How are these two quantities related?

5 Chapter 5: Three-dimensional euclidean theory

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6 Chapter 6: Bubble and the cosmological constant

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7 Chapter 7: The real world: four-dimensional lorentzian theory

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8 Chapter 8: Classical limit

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9 Chapter 9: Matter

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10 Chapter 10: Black holes

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11 Chapter 11: Cosmology

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12 Chapter 12: Scattering

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13 Chapter 13: Final remarks

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