Cosmological Physics

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Abstract

These are partial solutions for the textbook, Cosmological Physics by John A. Peacock. Before using these solutions, or referencing to them, please double-check them, and if an error or misunderstanding of the problem is found, please email me. I may not go through each chapter, since they may not pertain to what I want to conduct research in. My main focus will be on the theoretical developments within cosmology such as inflation, early universe dynamics, and perhaps one or two chapters on galaxy formation and clustering. These solutions are uploaded to my website: alexcassem.net on the blog tab.

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1 Chapter 1: Essentials of general relativity

Exercise 1.1: Use the idea of gravitational time dilation to explain why starts suffer no time dilation due to their apparently high transverse velocities as viewed form the frame of the rotating Earth.

Solution: This can be solved via the same mode of though as of page 7 when talking about the time dilation within a rocket. However, instead of using $v = gh/c$, we are now imagining the Earth and some star forms a cylinder with radius r, so we have a velocity of $v = \omega r$. We can plug this into the time dilation formula, $\Delta t' = \gamma \delta t$ to get a gamma factor of,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}} \simeq 1 + \frac{\omega^2 r^2}{2c^2},$$

where we used the binomial expansion formula to second order, $(1+x)^n \simeq 1+nx$, with $x = \frac{-\omega^2 r^2}{2c^2}$ and $n = -1/2$. But, this dilation is not observed, so what cancels it? Well, in order for stars to maintain approximately circular orbits, they must have a centripetal acceleration of $a = v^2/r$. Recall that the potential energy can be found from $\Phi = \int \mathbf{F} \cdot d\mathbf{r} \simeq r^2\omega^2/2$ which is found after integrating the force, mv^2/r and plugging in the value for v. This is a gravitational blueshift, which would cancel with the redshift we calculated above (at least towards second order).

You can also think of it as the light due to the large Lorentz factor redshifts the light as it approaches us, but due to the gravitational potential between the star and Earth, this is canceled out since our view from Earth is that of a rotating frame.

Exercise 1.4: Consider the path taken by an aeroplane in traveling between two points on the surface of the Earth, which should be an arc of a great circle for optimum fuel efficiency. Attempt to prove this by writing down the equation describing a geodesic on the surface of a sphere.

Solution: So, there are two methods to solving this problem. First, the classic general relativity way by taking the metric of a sphere, $g_{\mu\nu}(\theta, \phi) = r^2 * diag(1, sin^2(\theta))$, finding the Christoffel connections to use in the geodesic equation, solve two equations given via the geodesic equation for θ and ϕ, and determine the numerical factor needed to keep the plane on an optimum path via a great circle (I used $\theta \to \pi/2$ and found that $\phi(\tau, \theta) = exp(-2\cot(\theta)\tau)$ needs to be constrained). However, there is a much easier version if we simply solve the integral given by the line element,

$$ds^2 = r^2(d\theta^2 + sin^2(\theta)d\phi^2).$$

The setup then becomes,

$$s = r\int \frac{d\theta}{(1 + sin^2(\theta)\frac{d\phi}{d\tau})^{1/2}} := s(\phi', \phi, \theta; \tau).$$

This is in the form of a typical functional, which we can use Euler-Lagrange equations being

$$\frac{d}{d\tau}\frac{\partial s}{\partial \phi'} - \frac{\partial s}{\partial \phi} = 0.$$

Recall that $\phi' = \theta' =\neq 0$ but $r' = 0$. This gives the following equation of motion,

$$(1 + sin^2(\theta)\phi')^{1/2} \cdot sin(2\theta)\theta' + \frac{5}{2}(1 + sin^2(\theta)\phi')^{-5/2}(\phi'' sin^2(\theta) + sin(2\theta)\phi'\theta') = 0. \quad (1)$$

From this, the most trivial solution is when $\frac{d\phi}{d\tau} = 0$, in where this describes a plane following a great circle, or they never change the azimuth angle. However, to find anymore solutions without further constraints ($\theta = \pi/2$), this requires a lot of numerical computation.

Exercise 1.5: Show that the covariant equivalent of grad and curl ($A_{;\mu}$ and $B_{\mu;\nu} - B_{\nu;\mu}$) are the same in special relativity, but that the covariant divergence $C^{\mu}_{;\mu}$ is more complicated:

$$C^{\mu}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g} C^{\mu}).$$

Solution: For this problem, it is easiest to deal with by using the Principle of Equivalence which is a statement of any local inertial reference frame must be equal in a non-inertial frame. We can place ourselves in a local inertial frame (LIF), meaning all of our connections are zero, $\Gamma^{\mu}_{\nu\alpha} = 0$.[1] In this sense, the divergence becomes,

$$A_{;\mu} = \frac{D}{dx^{\mu}} = \frac{dA}{dx^{\mu}} + \cancelto{0}{\Gamma}, \tag{2}$$

which is just the usual gradient in special relativity if we restore the general basis,

$$\nabla A = \frac{\partial A}{\partial x^{\mu}} g^{\mu\nu} \mathbf{e}_{\nu}.$$

As for the curl, we can begin to compute it explicitly in a covariant frame, then impose our LIF,

$$\begin{aligned} B_{\mu;\nu} - B_{\nu;\mu} &= \frac{DB_{\mu}}{dx^{\nu}} - \frac{DB_{\nu}}{dx^{\mu}} \\ &= \frac{dB_{\mu}}{dx^{\nu}} + \Gamma^{\alpha}_{\mu\nu} B_{\alpha} - \left(\frac{dB_{\nu}}{dx^{\mu}} + \Gamma^{\beta}_{\mu\nu} B_{\beta} \right) \\ &= \left(\frac{dB_{\mu}}{dx^{\nu}} - \frac{dB_{\nu}}{dx^{\mu}} + \cancelto{0}{\Gamma^{\alpha}_{\mu\nu}} + \cancelto{0}{\Gamma^{\beta}_{\mu\nu}}, \right) \end{aligned} \tag{3}$$

which brings us back to the normal curl. However the divergence is a different story. In mathematical terms, everything we have written so far have been normalized to a unit one length, but for the divergence this can not be translated over easily from a LIF to a general covariant one. In general coordinates, the divergence of some vector $\mathbf{X}$ is given by $div(\mathbf{X}) = \nabla \cdot \mathbf{X} = X^{\mu}_{;\mu}$. This is just the d'Alembertian operator given by $\Box A \equiv (g^{\mu\nu} A_{;\nu})_{;\mu}$ which can be expressed as,

$$\Box A = g^{\mu\nu} A_{,\mu\nu} + \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{\mu\nu})_{,\mu} A_{,\nu}. \tag{4}$$

The other way to find this divergence is by computing the frame difference going from a general frame of reference to a Riemann one via, $div(\mathbf{X}) = (1/g)\partial(gX^{\mu})/\partial x^{\mu}$, where r is the local coefficient of the volume element. The volume element of a Riemann metric is of course $\sqrt{-g}$.

Exercise 1.6: By working in a local inertial frame, prove that Bianchi identity

$$(R^{\mu\nu} - g^{\mu\nu} R/2)_{;\nu}.$$

Solution: Just as the question suggests, working in the LIF makes this calculation easier, **and**, as a subtlety, this means we must require the Ricci scalar to be $R = 0$, since in any LIF we must recover special relativity, which is in flat space. So, we can just expand the covariant derivative to get,

$$\begin{aligned} (R^{\mu\nu} - g^{\mu\nu} R/2)_{;\nu} &= R_{\mu\nu;\nu} - (g_{\mu\nu} R/2)_{;\nu} \\ &= R_{\mu\nu,\nu} + \cancelto{0}{R_{\mu\alpha}\Gamma^{\alpha}_{\mu\nu}} + \cancelto{0}{R_{\beta\nu}\Gamma^{\beta}_{\mu\nu}} - g_{\mu\nu;\nu} - \frac{g_{\mu\nu}}{2} R_{;\nu} \\ &= g^{\mu\nu} R_{\mu\nu;\nu} - 2R_{,\nu} \\ R_{,\nu} - 2R_{,\nu} &= R_{,\nu} = 0, \end{aligned} \tag{5}$$

which is exactly what we stated we should see/are looking for above.

[1] This is an extremely useful tool for calculating physical quantities under a general relativistic framework.

2 Chapter 2: Astrophysical relativity

Exercise 2.1: The energy-momentum tensor $T_{\mu\nu} = (\rho + p/c^2)U^\mu U^\nu - pg^{\mu\nu}$ tells us that the energy flux density is $xT^{01} = \gamma^2(\rho c^2 + p)v$ and the momentum flux density is $T^{11} = \gamma^2(\rho + p/c^2)v^2 + p$, for a fluid moving at relativistic speeds. Use these relations to obtain the jump conditions for a relativistic shock and solve these in the case of (a) a strong shock propagating into a cold fluid; (b) a radiation-dominated fluid ($p = \rho c^2/3$).

Solution: Firstly, something that should be clarified is what a "jump condition" means since the text does not explain this. First, jump conditions are also called, Rankine-Hugoniot conditions which described the relationship between the states on both sides of a shock wave[2]. The specific *jump condition* holds at a discontinuity or abrupt change between two different mediums. The jump conditions in general come from the following DE,

$$\frac{d}{dt}\int_{x_1}^{x_2} w dx = -f(w)|_{x_1}^{x_2}, \tag{6}$$

where w is a conserved quantity. Equation 6 tells us how the conserved quantity behaves, while the next equation gives the actual jump condition for the conservation law,

$$u_s = \frac{f(w_1) - f(w_2)}{w_1 - w_2}. \tag{7}$$

The question is then, what conserved quantity do we use to describe a relativistic fluid? This will simply be the relativistic enthalpy, $w = \rho + p$ in units of $c = 1$. The conservation equation for enthalpy is equation 2.6 on page 36 namely,

$$\frac{d}{dt}\left(\frac{\gamma w}{n}\right) = \frac{\dot{p}}{\gamma n}, \tag{8}$$

with $\dot{p}$ being the time derivative of the pressure, γ is the usual Lorentz factor, and n is the particle density. The subscripts denote specific *flows* of the jump. Subscript 1 represents the jump conditions for just upstream, while the 2 represents the jump conditions for just downstream. To solve equation 8, we can let the Lorentz factor be $\gamma = cosh(\theta)$[3], and then solve in terms of hyperbolic coordinates. We get two equations, one from the LHS and the RHS when we integrate out namely,

$$\begin{aligned} w_1 sinh(\theta_1)cosh(\theta_1) &= w_2 sinh(\theta_2)cosh(\theta_2), \\ w_1 sinh^2(\theta_1) + p_1 &= w_2 sinh^2(\theta_2) + p_2. \end{aligned} \tag{9}$$

From equation 7, if we eliminate the downstream term (w_2), we can find the jump condition for upstream, and then switch the indices to find downstream,

$$\begin{aligned} u_1^2 &= \frac{(\rho_2 + p_1)(p_2 - p_1)}{(\rho_2 - \rho_1)(\rho_1 + p_2)} \\ u_2^2 &= \frac{(\rho_2 + p_2)(p_1 - p_2)}{(\rho_1 - \rho_2)(\rho_2 + p_1)} \end{aligned} \tag{10}$$

Then from equation 2.9 in the text, the equation of state reads,

$$p = (\Gamma - 1)(\rho - \rho_0) \rightarrow p = (\Gamma - 1)(\rho - n), \tag{11}$$

with n being the rest-mass density, $n \propto n/V$. The mass-density also has a conservation equation, $d/t(n\gamma) = constant$. With this, the jump conditions from equation 10, equation 9, and the conservation equation for mass density, we find the combination,

$$\frac{(\rho_2 + p_1)w_2}{n_2^2} = \frac{(\rho_1 + p_2)w_1}{n_1^2}. \tag{12}$$

[2]Information taken from Wiki found here.

[3]For why we can express γ this way click here.

Equation 12 can be hard to solve due to the hyperbolic equations, since the upstream and downstream equations of states differ. However, the question is asking for a strong shock case into a cold fluid. This can be found by taking the difference of Lorentz factors from downstream minus upstream which is,

$$\gamma_2 - \gamma_1 = \gamma_\Delta^2 = \frac{(\rho_1 + p_2)(\rho_2 + p_1)}{w_1 w_2}, \tag{13}$$

which is symmetric, meaning which ever way the shock travels, the shock condition above is the same. With this, and equation 12, we have the compression measure by a ratio of the densities,

$$\frac{n_2}{n_1} = \frac{\Gamma_2 \gamma_\Delta + 1}{\Gamma_2 - 1}. \tag{14}$$

For part (b) of this case, the factor $\Gamma = 4/3$ for upstream and downstream ($\Gamma_1 = \Gamma_2$) giving,

$$\gamma_1^2 = \frac{9}{8}(1 + \frac{\rho_2}{2\rho_1}). \tag{15}$$

Exercise 2.2: Use linearized gravity to obtain the metric inside a thin uniform shell of mass M and radius R, which rotates with angular velocity ω_{shell}. Show that the metric looks like a frame of reference that rotates at angular velocity,

$$\frac{\omega}{\omega_{shell}} = \frac{4GM}{3c^2 R}.$$

Solution: For this problem, we need to recall a few topics from shell theorem's, and perturbation theory from general relativity. We can either start by constructing the metric from geometric arguments, or by constructing the energy-momentum tensor by observation of the energy-momentum tensor for a collapsing star. Either way, the method works, it just depends upon one's comfort in either direction. For us, we can start from the metric approach.

For the metric, we will work in (+,-,-,-) notation (since it turns out easier after doing the calculation in the other notation). The metric for a general spacetime with spherical symmetry is,

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 sin^2(\theta). \tag{16}$$

For a spherical metric of some mass M, this mass in spacetime must have an angular velocity on the mass' edge. So, we can modify the metric as the following,

$$ds^2 = dt^2 - dr^2 - r^2 sin^2(\theta)(d\phi - \omega dt)^2 - r^2 d\theta^2. \tag{17}$$

This will be true for any spherically symmetric mass in spacetime that has some rotational velocity ω[3]. Now, we can construct the stress-energy tensor. First realize that θ and r are static components, and thus and term within $T_{\mu\nu}$ involving θ or r are zero. Also recall that $T_{\mu\nu}$ is symmetric, so we only need to find T^{00} and $T^{0\phi}$ components. The T^{00} is always the energy density, so in this case it will be the mass M divided by the surface area $4\pi R^2$ to give, $T^{00} = M/(4\pi R^2)$. However, we want to be able to measure at any r value inside this spherical mass, so we need a term to communicate this, which is the delta function to give (this is also something like a shell term),

$$T^{00} = \frac{M}{4\pi R^2}\delta(r - R). \tag{18}$$

For the "side" terms of the energy-momentum tensor, these are momentum density values, so we expected something like, $\sim mv/V$. The density term we can use from our T^{00} value, the v is $\Omega R sin(\theta)$, so we get,

$$T^{0\phi} = T^{\phi 0} = T^{00}(\Omega R sin(\theta)) = \frac{MR\Omega sin(\theta)}{4\pi R^2}\delta(r - R). \tag{19}$$

The question is asking for us to use linearized gravity to find a solution to first order, so we must use the Poisson equation being,

$$\nabla^2 h^{\mu\nu} = 16\pi G T^{\mu\nu}. \tag{20}$$

Our job becomes easier when we remember the question is asking for, "metric [that] looks like a frame of reference that rotates at angular velocity." This is just a dragged frame expressed by the angular velocity as, $\omega = h^{0\phi}/(rsin(\theta))$. So, we only need to solve for the $h^{0\phi}$, but this term is a potential in the view of the Poisson equation. Let this potential be ϕ. Then, we know we can express the second order differential equation in terms of spherical harmonics with only an ℓ term (since $d\theta = 0$, we have no m term) giving,

$$\phi'' + \frac{2}{r}\phi' - \ell(\ell+1)\frac{2}{r^2}\phi. \tag{21}$$

We want the lowest mode, so set $\updownarrow = 1$, and equate to the energy-momentum tensor,

$$\begin{aligned}\phi'' + \frac{2}{r}\phi' - 2\frac{2}{r^2}\phi &= 16\pi G * \left(\frac{MR\Omega}{4\pi R^2}\delta(r-R)\right), \\ \phi'' + \frac{2}{r}\phi' - 2\frac{2}{r^2}\phi &= \frac{4GM\Omega}{R}\delta(r-R)\end{aligned} \tag{22}$$

The interior solution goes as $1/r$ and the exterior solution goes as $1/r^2$. When we match $r = R$, and integrate the ϕ'' term, then we find,

$$\omega = \frac{4GM\Omega}{3R}, \tag{23}$$

where we are in units of $c = 1$, and $\Omega \to \omega_{shell}$. As a hint for the integral, you can reason the $1/r^2$ terms drops off fast, so we don't need it, and then combine the first and second term into, $\int d^3(\phi' r)$.

3 Chapter 3: The isotropic universe

Exercise 3.1: Show that the Robertson-Walker metric satisfies Einstein's equations and obtain the Friedmann equation(s).

Solution: There are two ways to solve this problem; either with *Mathematica* and using your favorite tensor package (mine is *diffgeo.m* found here), or via the brute force way, but being clever about indices. For people who are wondering how to use the *Mathematica* package, I have a video on my website on how to use the package. Otherwise, we will use the second method, good-ole' tensor calculus.

The metric for Robertson-Walker is,

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2(\theta)d\phi^2\right)\right]. \tag{24}$$

If we set, $R(t)/R_0 \equiv a(t)$, then the metric/line-element becomes,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-R_0^2kr^2} + r^2\left(d\theta^2 + sin^2(\theta)d\phi^2\right)\right]. \tag{25}$$

The typical Einstein equations can have their second index raised to make calculations easier which appears as,

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}\right)g^{\mu\alpha} \to R^\alpha_n u - \frac{1}{2}\delta^\alpha_n uR = T^\alpha_n u. \tag{26}$$

The stress-energy tensor for our system (universe) is for dust,

$$\left(T^{\mu\nu} = (\rho+p)U^\mu U^\nu - pg^{\mu\nu}\right)g_{\mu\alpha} \to T^\nu_\alpha = (p+\rho)U_\alpha U^\nu = p\delta^\nu_\alpha. \tag{27}$$

If we choose a specific frame of reference, say in the rest frame where $U_\alpha = (1, 0, 0, 0)$, then the stress-energy tensor becomes,

$$T_\alpha^\nu = diag(\rho, -p, -p, -p). \tag{28}$$

The Ricci tensor components will only be tt and the rr components due to spherical symmetry. These components are then,

$$\begin{aligned} R_t^t &= 3\frac{\ddot{a}}{a} \\ R_r^r &= \frac{2kR_0 + 2\dot{a}^2 + a\ddot{a}}{a^2} \end{aligned} \tag{29}$$

while the Ricci scalar will be,

$$R = \frac{6(kR_0 + \dot{a}^2 + a\ddot{a})}{a^2}. \tag{30}$$

Now, if we put together first the tt components, we get something like a "velocity" equation of the universe,

$$\begin{aligned} &R_t^t - \frac{1}{2}\delta_t^t R = T_t^t \rightarrow \\ &\frac{\ddot{a}}{a} - \frac{3(kR_0 + \dot{a}^2 + a\ddot{a})}{a^2} = \frac{\rho}{3} \\ &\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} + \frac{kR_0}{a^2}, restore\ units \rightarrow \\ &\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{kR_0c^2}{a^2}. \end{aligned} \tag{31}$$

Now for the rr equation, or an "acceleration" equation of the universe is,

$$\begin{aligned} &R_r^r - \frac{1}{2}\delta_r^r = T_r^r \rightarrow \\ &\frac{2kR_0 + 2\dot{a}^2 + a\ddot{a}}{a^2} - \frac{3(kR_0 + \dot{a}^2 + a\ddot{a})}{a^2} = -p \\ &-p + \frac{kR_0}{a^2} = -\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\ddot{a}}{a}, plug-in\ eq.31 \rightarrow \\ &\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(p - \rho) - \frac{R_0 k}{a^2}. \end{aligned} \tag{32}$$

The end of equation 31 and 32 are classically called the Friedmann equations.

Exercise 3.4: Consider the second-order corrections to the relation between redshift and angular-diameter distance $(c/H_0)D(z)$:

$$D(z) \simeq z - \frac{3 + q_0}{2}z^2 \Rightarrow z \simeq D(z) + \frac{3 + q_0}{2}D(z)^2, \tag{33}$$

to second order. Attempt to account for this relation with a Newtonian analysis.

Solution: The most direct way to explain this is first by recalling the scale factor, $a(t) = R(t)/R_0 \rightarrow R(t) = R_0 a(t)$, from which we can perform a Taylor series to get,

$$R(t) \simeq R_0 - R_0 H_0 t - \frac{R_0 q_0 H_0^2 t}{2}, \tag{34}$$

where $q_0 = -\ddot{R}R_0/\dot{R}$ is the deceleration parameter. Now, since $1 + z = a(t)/a_0$, this becomes $1 + z = R_0/R$, from which if we divide out the common R_0 term from equation 34, put this in terms of the integral of equation 3.39 (page 76) by recalling equation 3.22 relationship between distance and time for photons on a geodesic in Friedmann-Walker universe,

$$r = \int \frac{cdt}{R(t)}, \tag{35}$$

this gives the following integral,

$$D(z) \simeq \int (1 - (3 + q_0)z)dz \tag{36}$$

will give us the relationship as desired above. This states that physical objects are brighter when the parameter q_0 is larger, meaning that the radial acceleration is faster with respect to the velocity squared, or simply, the deceleration parameter has left the stars closer.

4 Chapter 4: Gravitational lensing

Exercise 4.1: Give a simple proof that surface brightness is conserved in Euclidean space. Now show that volume elements in phase space are Lorentz invariant, and hence that the relativistic expression of surface-brightness conservation is that I_v/v^3 is an invariant. Show from this that black body radiation appears thermal to all observers. If this is so, how is it possible to use the microwave background to determine that the Earth has an absolute velocity of $\simeq 370 km\, s^{-1}$.

Solution: The first part is showing that the surface brightness (flux through a surface, which has units of brightness (B) times the area (A_n) times the solid angle (Ω_n)). So, consider a source of area A_1 with solid angle Ω_1. Then the detector has a solid angle of Ω_2 with respect to the source. The total flux through the detector, $\Phi_d = BA_d\Omega_e$ ($\Phi \sim$ what it accepts times the angle the detector shows). Then, the intensity is,

$$I = \frac{\Phi}{A_d\Omega_e} = \frac{BA_e\Omega_d}{A_d\Omega_e}, \tag{37}$$

and if this is to be conserved, $I = 1$, giving,

$$\frac{A_e}{A_d} = \frac{\Omega_e}{\Omega_d}. \tag{38}$$

So, suppose $\Omega = A/r^2$, then this quantity is always true given the relationship between radii. The second part of the question is showing the volume element for an intensity by photons is Lorentz invariant. Suppose we are only boosting along the x-axis, then $(dp_ydydp_zdz)' \to dp_ydp_zdydz$, since the volume element is in phase-space. We only need to focus on dx and dp_x. Recall that for position (think length contraction) goes, $dx \to \gamma^{-1}dx$ and the momentum transforms as $dp_x \to \gamma dp_x$. Then, $(dp_xdx)' \to \gamma\gamma^{-1}dp_xdx = dp_xdx$. Thus, the phase-space volume element/density is invariant.

Now for the surface-brightness conservation is invariant, this is also just the number density of photons which is the flux-energy density divided by c and the photon energy $\hbar\omega$ times dV (the volume element above). This becomes,

$$N = \frac{I_\omega d\Omega d\omega dV}{\hbar\omega c} = \frac{I_\omega d\Omega\omega^2 d\omega dV}{\hbar\omega^3 c}. \tag{39}$$

If we do a Fourier transform to momentum space, the differential becomes, $d^3p = (\hbar^3/c^3)d\Omega d\omega$; so to keep this and the above invariant, we need to require the phase-space density to be $c^2 I_\omega/(\hbar^4\omega^3)$. This is just black body radiation, $I_\omega \propto \omega^3\bar{n}$, with the $\bar{n}$ the occupation number depicted by the typical distribution, $\bar{n} = (exp(\hbar\omega/kT) + 1)^{-1}$.

Finally, the third part of the question is asking how is it possible to use the CMB to determine the Earth has an absolute velocity. Well, we have to scale the frequency with respect to the temperature to give,

$$\frac{T'}{T} = \frac{\omega'}{\omega}. \tag{40}$$

This way, the CMB still appears as thermal radiation. Thus, the CMB dipole does not prove absolute motion, and is interpreted as only a velocity subject to the condition the background of the CMB is isotropic.

5 Chapter 5: The age and distance scales

Exercise 5.1: Show that, for a star in which the opacity is due to Thomson scattering, the luminosity, temperature and size are expected to scale as follow: $L \propto M^3, T \propto M^{1/2}, R \propto M^{1/2}$, independently of the detailed mechanism for central energy generation.

Solution: We can use the system of a uniform density star. By the Virial theorem, $T \sim V$, giving $GM^2/R \sim nk_BT$ giving $T \propto M/R$. To figure out the luminosity and radius (size), we can think of the mean-free path of photons inside the star as $\lambda \propto R^3/M$, which has a time to diffuse a distance R given by, $t \sim R^2/c\lambda = M/Rc$. The luminosity is in energy of units given by, $E \propto R^3T^4$, divided by the time it takes for photons to escape, $L \propto RT^4\lambda \propto R^4T^4/M$. And via the Virial theorem, this becomes, $L \propto M^3$. We can also say that since the star is like a blackbody, $L \propto R^2T^4$, we can use the relationship for temperature and energy above to find, $T \propto M^{1/2}$ and $R \propto M^{1/2}$.

6 Chapter 6: Quantum mechanics and relativity

Exercise 6.1: Show that the nonrelativistic Schrödinger equation can be manipulated into the form of a conservation law $\dot{\rho} + \nabla \cdot \mathbf{j} = 0$ where $\rho = |\psi|^2$ is thus to be interpreted as the conserved density of charge, mass, or probability. What is the expression for the current $\mathbf{j}$?

Solution: Recall that the Schrödinger equation, but in a different orientation,

$$i\hbar\dot{\psi} - H\psi = 0, \; with \; H = -\frac{\hbar^2}{2m}\nabla^2 + V. \tag{41}$$

We are told to put this in terms of $\rho = |\psi|^2 = \psi\psi^*$, so lets find the complex conjugate of equation 41,

$$-i\hbar\dot{\psi}^* - \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi^* = 0. \tag{42}$$

We can then multiply to equation 41 ψ and ψ^* to equation 42 to get like terms if we subtract them from each other. This step can be seen by the need to get derivative products of both ψ and ψ^* in order to bring them together as $\partial_t(\psi^*\psi)$ which is shown below,

$$\begin{aligned} \psi S^* - \psi^* S &= 0, \\ \left(-i\hbar\psi\dot{\psi}^* - i\hbar\psi^*\dot{\psi}\right) + \frac{\hbar^2}{2m}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) &= 0, \\ -i\hbar\left(\psi\dot{\psi}^* + \psi^*\dot{\psi}\right) &= -\frac{\hbar^2}{2m}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) \\ \frac{\partial}{\partial t}|\psi|^2 &= -\frac{\hbar^2}{2m}\nabla\cdot\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right), \\ \frac{\partial\rho}{\partial t} &= -\nabla\cdot\mathbf{j}, \end{aligned} \tag{43}$$

where the ρ is just as defined above, and all left is to add over the term on the right to get the desired answer. The current, from our derivation, must then be,

$$\mathbf{j} = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right). \tag{44}$$

Exercise 6.2: Show that the 4-current $J^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$ is conserved for a field that obeys the Klein-Gordon equation. Show that plane-wave solutions to the Klein-Gordon equation have $E = \pm\sqrt{k^2+m^2}$ and that the negative-energy solutions have negative J^0. If this is not a probability density, what is being conserved here?

Solution: Recall that the Klein-Gordon equation (KG) is,

$$\left(\partial^\mu\partial_\mu + \frac{m^2c^2}{\hbar^2}\right)\phi = 0, \tag{45}$$

but many times we will suppress either $\hbar = c = 1$. Just like last problem, we want to put this in terms of the 4-current, but now, $\dot{\rho} + \nabla\cdot\mathbf{j} = \partial_\mu J^\mu = 0$ given that $J^\mu = (c\rho, \mathbf{j})$. We could actually guess what the relativistic probability current should look like given that we completed the last problem, plug this into the conservation equation, and see what comes out. We can guess the current to be,

$$J^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right). \tag{46}$$

If we now take this and plug into equation 45 we find,

$$\begin{aligned}\partial_\mu\left(i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)\right) &= 0\\ i\partial_\mu\phi^*\partial^\mu\phi + i\partial_\mu\partial^\mu\phi(\phi^*) - i\partial_\mu\phi\partial^\mu\phi^* - i\partial_\mu\partial^\mu\phi^*(\phi) &= 0\\ i\partial_\mu\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) &= 0\\ \partial_\mu J^\mu &= 0.\end{aligned} \tag{47}$$

One can pick out that our guess in not entirely correct, since when using the KG equation in line 3 of equation 47, we would need to redefine our current by multiplying by a factor of $\hbar/2m$ to get the algebra to work.

We now check that a plane-wave solution gives the energy above with negative-energy solutions a negative current J^0. Consider the general plane-wave solution,

$$\phi(x^\mu) = \phi(\mathbf{r})e^{-iEt/\hbar} = Ae^{i\mathbf{k}\cdot\mathbf{x} - iEt/\hbar}, \tag{48}$$

where the energy is $E = \hbar\omega$, and A is some normalization constant. If we plug this into the KG equation we get,

$$\begin{aligned}\partial^\mu\partial_\mu\left(Ae^{i\mathbf{k}\cdot\mathbf{x} - iEt/\hbar}\right) + m^2\phi &= 0\\ \partial^\mu\left(Ae^{i\mathbf{k}\cdot\mathbf{x} - iEt/\hbar}(i\mathbf{k} - iE/\hbar) + m^2\phi\right) &= 0\\ \left(-E^2 + k^2 + m^2\right)\phi &= 0 \rightarrow\\ E &= \pm\sqrt{k^2+m^2}.\end{aligned} \tag{49}$$

The final question is what the current is for J^0 for negative energy states. Well, given that $\partial^\mu\phi = (\dot{phi}, -\nabla\phi) = (-iE, -i\mathbf{k})\phi$, the current is simply, $J^0 = 2E|\phi|^2$.

Exercise 6.3: Define the density matrix ρ as,

$$\rho \equiv \sum_i p^i \left|i\right\rangle \left\langle i\right|,$$

where p_i is the probability that the system under study has been prepared in the state $|i\rangle$. Show that the expectation of an operator A is given by $Tr(A\rho) = \sum_n \langle n| A\rho |n\rangle$, and hence that the density matrix for a system in thermal equilibrium is $\rho \propto exp(-H/kT)$.

Solution: We can just being to plug in ρ into the expectation value we want and see if the trace comes out,

$$\begin{aligned}Tr(A\rho) &= \sum_n \langle n| A\rho |n\rangle \\ &= \sum_{i,n} p_i \langle n| A |i\rangle \langle i|n\rangle \\ &= \sum_{i,n} p_i |n\rangle \langle n| \langle i| A |i\rangle \\ &= \sum_i \langle i| A\rho |i\rangle .\end{aligned} \tag{50}$$

To show that the density of a system in thermal equilibrium equals that of the above, we can do a makeshift construction of a quantum-statistical partition function as so,

$$\begin{aligned}Z_i &= \sum_i e^{-E_i/k_BT} \\ &= \sum_i e^{-E_i/k_BT} |i\rangle \langle i| \rightarrow \\ recall\ &: e^{E_i/k_BT} |i\rangle = e^{-H/k_BT} |i\rangle ,\ plug-in \\ &= \sum_i e^{-H/k_BT} \langle i|i\rangle \\ &= \sum_i p_i |i\rangle \langle i| .\end{aligned} \tag{51}$$

Exercise 6.4: Define the entropy of a quantum-mechanical system in terms of the occupation probabilities of its microstates:

$$S = -k \sum_i p_i ln(p_i).$$

Use the golden rule and first-order perturbation theory to show that this quantity always increases with time. How is this possible, given that the nonrelativistic Schroödinger equation is symmetric under time reversal?

Solution: By Golden Rule, the text refers to Fermi's rule for the density of transition between states for initial to final, $i \rightarrow f$,

$$\lambda_{if} = \frac{2\pi}{\hbar} |M_{if}|^2 \rho(E_f) = \frac{2\pi}{\hbar} | \langle f| \Delta H |i\rangle |^2 \rho_f. \tag{52}$$

This is only a one way street since the underlying assumption is we go from $p_i = 1$ to $p_f = 0$ (or n-1 transition). To make it a two-way street, we can multiply by $p_i - p_f$. For states that are at equilibrium, $p_i = p_f$, the entropy becomes,

$$S = -k_B \sum_i p_i ln(p_i) = -k_B(p_1 ln(p_1) + p_2 ln(p_2) + ...) = -k_B p ln(p!) = k_B ln(W), \tag{53}$$

given that $W = p!/(\prod p)!$. If we go out of phase, the entropy changes as $dS/dp_i = dS = k_B ln(p_f/p_i)dp_i$. From this, $dp_i = -dp_j$ which if we compare to the golden rule is $\propto (p_f - p_i) \rightarrow dS > 0$. The reason why this is forward in time, and does not have the time-reversal symmetry of the Schrödinger equation is because the general assumption when deriving the golden rule is that the perturbation begins at $t = 0$, and the states afterwards have no "memory" of the previous states.

Exercise 6.6: Show that the time-reversal operator for the Dirac equation is $T\psi = \gamma^1\gamma^3\psi^*$ in the Dirac representation. Defining $P\psi = \gamma^0\psi$ and $C\psi = \gamma^2\psi^*$, find the matrix corresponding to CPT

and show directly that the Dirac equation is invariant under this operation together with $x^\mu \to -x^\mu$ and $e \to -e$.

Solution: Recall the Dirac matrices in the Dirac representation, $\gamma^0 = diag(1,1,-1,-1)$. $\gamma^1 = antidiag(1,1,-1,-1))$, $\gamma^2 = anitdiag(-i,i,i,-i)$, and γ^3 is,

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Also recall that the Dirac matrices are anticommuting, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$. To check for parity invariance under the parity operator, $P\psi = \gamma^0\psi$, we can split up the Dirac equation with an $U(1)$ gauge field into its time and spatial parts,

$$\begin{aligned} (\gamma^\mu(P_\mu - eA_\mu) - mc)\psi = 0 \\ (\gamma^0(P_0 - eA_0) + \gamma^i(P_i - eA_i) - mc)\psi = 0 \end{aligned} \tag{54}$$

To check the parity, we flip the sign on the spatial gamma matrices and P_0, and apply the parity operator,

$$\begin{aligned} (\gamma^0(-P_0 - eA_0) - \gamma^i(P_i - eA_i) - mc)P\psi = 0, \\ (\gamma^0(-P_0 - eA_0) - \gamma^i(P_i - eA_i) - mc)\gamma^0\psi = 0 \\ (\gamma^0(-P_0 - eA_0)\gamma^0 - \gamma^i\gamma^0(P_i - eA_i) - mc)\psi = 0 \\ (\gamma^0(P_0 - eA_0) + \gamma^i(P_i - eA_i) - mc)\psi = 0, \end{aligned} \tag{55}$$

since any multiple of γ^0 will change the parity, and we can simply insert the anticommuting relationship, and then pull out the γ^0 factor after multiplying through.The same thing occurs with the charge operator since $\gamma^2\gamma^{\mu*} = \gamma^2\gamma^\mu = -\gamma^\mu\gamma^2$ since the third matrix is complex. So, when we flip the sign of the charge e and then take the complex conjugate, the momentum changes $P_\mu \to -P_\mu$ since $P_\mu = i\hbar\partial_\mu$, the sign of the charge is taken care of by the anticommuting relationship, and then if we multiply through the γ^2 term on P_μ, we recover P_μ. The same thing happens for the T operator but with the relationship, $\gamma^1\gamma^3\gamma^\mu = -\gamma^\mu\gamma^1\gamma^3$, since we need to only change the temporal part of $P_\mu \simeq (E/C, p_i)$.

The overall Dirac equation is invariant under $CPT\psi$ since all we have to do is plugin each individual operator,

$$CPT\psi = CP(\gamma^1\gamma^3\psi^*) = C(\gamma^0\gamma^1\gamma^3\psi^*) = \gamma^2\gamma^0\gamma^1\gamma^3\psi, \tag{56}$$

where the ψ changes its conjugation since the conjugate of a conjugate (first T then C) is just the real. Recall from page 163 that $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. To get our value in terms of γ^5, we can use the anticommuting rule on first the 0 and 2 gamma matrix and pick-up a negative, and then again on the 2 and 1 matrix to get another negative as so,

$$\gamma^2\gamma^0\gamma^1\gamma^3 = -\gamma^0\gamma^2\gamma^1\gamma^3 = -(-)\gamma^0\gamma^1\gamma^2\gamma^3. \tag{57}$$

To have it match γ^5, multiply by i to both and divide out the negative to get the matrix for the CPT operator, $-i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$. Which, if we multiply through the Dirac equation **after** changing the sign of the parity, charge, and time signs.

7 Chapter 7: Quantum Field Theory

Equation 7.1: Show that the first-order perturbation term for quantum mechanics with an electromagnetic field, $(e/m)\mathbf{A}\cdot\mathbf{p}$, is proportional to the electric dipole moment. What is the interpretation of the A^2 term?

Solution: This one is a little tricky since it is based on what/how we construct the Hamiltonian. Consider the Hamiltonian in equation 6.36 give by,

$$H \equiv \mathbf{p} \cdot \mathbf{v} - L = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - |p|^2 + e\phi + V. \tag{58}$$

The $|p|^2$ term comes from $-m^4c^2$. In order for us to get the correct dipole moment, $\phi = V = 0$. Now, if we expand out the Hamiltonian we get,

$$H = \frac{1}{2m}(|\not{p}|^2 - |\not{p}|^2 - e\mathbf{A} \cdot \mathbf{p} - e\mathbf{p} \cdot \mathbf{A} + e^2|A|^2). \tag{59}$$

Obviously, the first two terms cancel out, the second term stays, the third goes something like, $\mathbf{p} \cdot \mathbf{A} \simeq \nabla \cdot \mathbf{A}$, which if we choose the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$, page 177), then this is also zero, and the last term of course stays. To first order then, the perturbation gives $-e\mathbf{A} \cdot \mathbf{p}$. To decide if this is a dipole or not, we can derive the matrix perturbation from $\langle i| \Delta H |j\rangle = ei\mathbf{A} \cdot \mathbf{r}\Delta E/\hbar$. To see how this comes about, the commutation of the Hamiltonian with the position gives, $[H, \mathbf{r}] = -i\hbar\mathbf{p}/m$, which can be used when expanding the states ψ_i and ψ_j to bring together commutator relationships. This cancels out the $\mathbf{A} \cdot \mathbf{p}$ term and we get the relationship with position above.

To simplify this further, we can argue that from the relationship, $\Delta E \Delta t \simeq \hbar$ gives, $\Delta E/\hbar \simeq 1/\Delta t = \omega_{ij}$. The time-derivative of the vector potential is, $\dot{A} = -i\omega A$, but this in the Coulomb gauge is simply, $\mathbf{E} = \dot{\mathbf{A}}$. The perturbation then becomes after plugging in and simplifying,

$$ei\mathbf{A} \cdot \mathbf{r}\Delta E/\hbar = ei\mathbf{A} \cdot \mathbf{r}\omega_{ij} = -e\dot{\mathbf{A}} \cdot \mathbf{r} = -e\mathbf{E} \cdot \mathbf{r}, \tag{60}$$

which is exactly the electric dipole moment. We can interpret the $|A|^2$ term as the square of a transition amplitude of a photon. So, it is a two-photon emission from the view of field theory.

Exercise 7.3: Consider a set of classical Grassmann variables, which are assumed to anticommute: $\{x_i, x_j\} = 0$. Show that derivatives with respect to these variables also anticommute, and that integration is identical to differentiation for Grassmann variables.

Solution: Recall that Grassmann numbers are complex variables that describe the geometry of a complex manifold (used in exterior algebra). So, to show that the integration over a Grassmann number is the same as the derivative, consider a general Grassmann number, $x = \rightarrow x + x_0$. Then the integration over this is,

$$\int dx(bx + ax_0) = \int dx\, x + \int dx\, x_0 = \int dx\, x + 0, \tag{61}$$

with a, b constants, and in the last step is a fundamental property of Grassmann numbers. And if we put a normalization on the integral, $\int dx\, x = 1$, then the integral in equation 61 is just b. Now, consider the derivative,

$$\frac{\partial}{\partial x}(ax_0 + bx) = \frac{\partial}{\partial x}(ax_0) + \frac{\partial}{\partial x}(xb) = b, \tag{62}$$

which is the exact same as the integral. To show that the derivative commutes, we only need to understand the commutator in the question when we set $i = j$ which gives $(x_i)^2 = 0$. Now, consider the expansion of a Grassmann number, ax,

$$ax \simeq= 1 + ax_i + a\frac{dx_i}{dx} + a\frac{d^2x_i}{dx^2} + a\frac{d^3x_i}{dx^3}... \tag{63}$$

But, the second term must be zero, since if we are to have a second term, it will be of the form x_i^2 which is zero from the commutator. So the expansion will only have first order terms, and if the variation is zero, then the derivative of the commutator is also 0.

Exercise 7.5: By using dimensional arguments or otherwise, show that the contribution to the vacuum energy density due to virtual particles of mass m must be, $\rho_{vac} \sim m^4c^3/\hbar^3$.

Solution: Consider the uncertainty principle for energy, $\Delta E\Delta t \simeq \hbar$. A virtual pair of particles will have energy $E \sim mc^2$, which if we plug in gives, $t \sim \hbar/(mc^2)$. If we divide out one of the $c's$, we get,

$$tc \sim \frac{\hbar}{mc} \sim x, \tag{64}$$

since this is how position is defined in spacetime. Now, since density is about $\rho \sim mass/volume \sim m/x^3$, which if we plug in for x gives,

$$\rho \sim \frac{m}{(\hbar/mc)^3} \sim \frac{c^3m^4}{\hbar^3}. \tag{65}$$

If we use the value for the Planck mass given in the front of the book, $m_p \sim 2.177e-8$, will give a vacuum density estimation of, $\rho \sim 10^{96.71}kg$, which exceed the cosmological density presented in chapter 4 by a factor of $10^{122.44}(\Omega h^2)^{-1}$.... which is why everyone is worried.

Exercise 7.6: Derive Feynman's expression for the path integral that gives the propagator in nonrelativstic quantum mechanics.

Solution: This is a very common derivation which can be found in multiple sources, such as, Introduction to Quantum Mechanics, by Griffith. In this text, I will re-derive it, but add a few details here and there that are missed.

The goal is to find equation 7.120 which is,

$$\langle q', t'|q, t\rangle \propto \int \mathcal{D}q\mathcal{D}p\, exp\left[\frac{i}{\hbar}\int_t^{t'} L(q,\dot{q})dt\right]. \tag{66}$$

Now, the text does give an answer for this question, but it is not that helpful, at all. The book's version started with the 1D propagator, and then introduces a lot of terms and no details. We will start with the unitary time-operator, $\hat{U}$, and then construct up to n number of paths/propagators to get the generalized path integral above. So, recall that the time-operator is,

$$\hat{U}(t''-t') = \left(e^{-i\hat{H}\Delta t}\right)^n, \tag{67}$$

where we are take n products of operators corresponding to the number of *times* a particle propagates. We can insert a complete set of operators (just like a 1), $\mathbb{1} = \int dq\,|q\rangle\langle q|$, and then this same term but for $n-1$ terms,

$$\langle q''|\,\hat{U}(t''-t')\,|q'\rangle = \int dq_1 \int dq_2 ... \int dq_{n-1}\,\langle q_n|\,e^{-i\hat{H}\Delta t}\,|q_{n-1}\rangle\,\langle q_{n-1}|\,e^{-i\hat{H}\Delta t}\,|q_{n-2}\rangle\,...\,\langle q_1|\,e^{-i\hat{H}\Delta t}\,|q_0\rangle\,. \tag{68}$$

Since there are n number of terms, we will have n number of delta functions, and an n number of differentials/integrals when evaluating each propagator if we recall,

$$\langle q_{n+1}|q_n\rangle = \frac{1}{2\pi}\int exp\left(ip(q_{n+1}-q_n)\right),\ and\ \langle q_{n+1}|\,\hat{H}\,|q_n\rangle = \frac{1}{2\pi}\int exp\left(ip(q_{n+1}-q_n)H(p,q)\right). \tag{69}$$

However, there are n summations of the position and momentum within the exponential, so all together we will get,

$$\langle q''|\,\hat{U}(t''-t')\,|q'\rangle = \int \frac{dp_n}{2\pi}\prod_{k=1}^{n-1}\frac{dp_kdq_k}{2\pi}exp\left(i\sum_{k=1}^{n}[p_k(q_k-q_{k-1})-H(p_k,q_{k-1})\Delta t]\right), \tag{70}$$

where $H(p,q)$ is now the classical Hamiltonian. Looking back at equation 66 above (7.120 in the text), we need to take the limit in our exponential, $\lim_{n\to\infty}$, to get the integral, which makes the variables p_k and q_k become dependent upon time. This means we also have to take the same limit on the differentials giving us,

$$\langle q''|\,\hat{U}(t''-t')\,|q'\rangle = \int \frac{dp(t)}{2\pi}\prod_t \frac{dp(t)dq(t)}{2\pi} exp\left(i\int dt[p(t)\dot{q}(t) - H(p,q)]\right). \tag{71}$$

The differential in front was only there since there was $n-1$ terms to integrate over. However, since those are now gone thanks to the limit, this differential can simply go away giving,

$$\langle q''|\,\hat{U}(t''-t')\,|q'\rangle = \int \prod_t \frac{dp(t)dq(t)}{2\pi} exp\left(i\int dt[p(t)\dot{q}(t) - H(p,q)]\right). \tag{72}$$

Still, we don't have equation 66's form, until we consider two things. First, we can define a *measure*, which describes how we calculate length or momentum within the integral/space, being $\mathcal{D}q = \prod_t dq(t)$ and $\mathcal{D}p = \prod_t dp(t)$. Second, recall that the Hamiltonian is normally defined as $H(p,q) = pv - L = p(t)\dot{q}(t) - L$. From these two pieces of information, we can sub them into equation 72 to get,

$$\langle q', t'|q, t\rangle = \frac{1}{2\pi}\int \mathcal{D}q\mathcal{D}p\, exp\left[i\int_t^{t'} L(q,\dot{q})dt\right], \tag{73}$$

where we reabsorbed the explicit unitary operator, and if you restore units you recover the $\hbar$ as in equation 66.

8 Chapter 8: The standard model and beyond

Exercise 8.1: Derive the Noether current and energy-momentum tensor for a Lagrangian that is invariant under internal and spacetime translation symmetry.

Solution: The text does not go into detail at all on how to find the two currents for the symmetries asked for, so I will attempt to clear up what equation 8.6 and 8.7 on page 217 and page 218 mean/are describing. Then, I will give the basic definition of the stress-energy tensor from equation 8.6.

For an internal symmetry, we can look at a complex-scalar field for a simple example. This type of Lagrangian has the variables, $\mathcal{L} = \mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*)$. If we transform the field-variables as,

$$\begin{aligned} \phi(x) &\to \phi'(x) = e^{i\alpha}\phi(x) \\ \phi^*(x) &\to \phi'^*(x) = e^{-i\alpha}\phi^*(x), \end{aligned} \tag{74}$$

then the Lagrangian must also be invariant $\mathcal{L} \to \mathcal{L}'$. Now, if we use Hamilton's principle, $\delta\mathcal{L} = 0$, the transformation of the field must also be varied, $\delta\phi(x) = i\alpha\phi(x)$ and similar for the complex conjugate. The total variation of the Lagrangian is then,

$$\delta\mathcal{L}' = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*). \tag{75}$$

If we use the Euler-Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right), \tag{76}$$

then we can eliminate the terms with respect to ϕ and ϕ^*, and gives us the product rule for the rest of the terms to give,

$$\delta\mathcal{L}' = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*\right] = 0. \tag{77}$$

And by Hamilton's principle, since the entire variation must be 0, so must the inside, thus if we define the inside of the partial derivative as,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^*, \tag{78}$$

we get the conservation equation, $\partial_\mu j^\mu = 0$, where $j^\mu(x)$ is defined as Noether's current. And if we want the explicit form, we can plug in the variation of ϕ and ϕ^*. (We will hold off on the stress-energy tensor).

Now, for the spacetime translation symmetry, take a look at the constant we put into our complex scalar transformation α; this is simply a number **not** a function. However, if we let $\alpha \to \alpha(x)$, then it becomes a spacetime translation symmetry (or a local transformation, the above is called a global transformation).

Now, I will also remind the reader that we *could* have a set of scalar fields at one point, which can be denoted as ϕ_i. To take a look at this type of symmetry and find a general current, we won't consider a complex scalar, just a normal one, so there is only one transformation equation,

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \delta\phi_i, \tag{79}$$

where the variation with respect to the spacetime symmetry is,

$$\delta\phi_i = \epsilon(x) f_i(x), \tag{80}$$

given some function that is dependent upon the local spacetime coordinates x^μ, and f_i is the transformed scalar function.

Now, we do the same as we did above with variation of the Lagrangian, identify the current, and simplify. There will be one more step though with the first term in the expansion, but after an integration by parts, this term goes to zero by Euler-Lagrange equations:

$$\delta\mathcal{L} = \epsilon\left[\frac{\partial \mathcal{L}}{\partial \phi_i} f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\mu f_i\right] + \epsilon\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} f_i\right]. \tag{81}$$

As we said previously, if you integration by part on the first term to get rid of the $\partial_\mu f_i$ which then gives the Euler-Lagrange equations, which is zero. The second term is our current,

$$j^\mu(x) = \epsilon\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} f_i, \tag{82}$$

and thus must be conserved as we can show,

$$\begin{aligned}\partial_\mu j^\mu(x) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} f_i\right) &= f_i\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)i}\partial_\mu f_i \\ &= \frac{\partial \mathcal{L}}{\partial_\mu \phi_i} f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\mu f_i.\end{aligned} \tag{83}$$

And finally, for the final part of the question, we can find the stress energy tensor in a direct manner (although it won't be symmetric) via,

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \mathcal{L}\delta^\mu_\nu, \tag{84}$$

where the $\partial_\nu\phi \equiv \partial_\nu\phi$, is the derivative of the symmetry change as we define, for example, in equation 80 which came from equation 79.

Exercise 8.3: Why, when you look in a mirror, did you appear with left and right sides transposed, but not upside-down?

Solution: This is most useful if you have an *actual* mirror in your hands to work with. We can think of the mirror as a group itself with elements. So, what could we do with a mirror? The mirror has the following group elements, M_1, M_2, R, and I. We can then think of an inversion, a left-right transformation as $-M_2 = M_1$, which from the group table is equivalent to, $M_1 \circ R = M_2$. Thus a reflection is not distinguishable from a rotation of the second half of the mirror. So we could be upside down, but then we would never know since we would be rotated and look exactly like reflection.

Exercise 8.4: In the electroweak model, the Higgs potential is,

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2. \tag{85}$$

Show that the potential minimum can always be chose as $\langle 0 | \phi | 0 \rangle = (0, v/\sqrt{2})$. Is this choice consistent with the numbers of degrees of freedom in the model?

Solution: From the above potential, to find the minimum simply solve,

$$\begin{aligned} \frac{dV}{d\phi} &= \mu^2 + 2\lambda(\phi^\dagger \phi) = 0, \\ \phi^\dagger \phi &= \frac{-\mu^2}{2\lambda} = \frac{v^2}{2}, \end{aligned} \tag{86}$$

if we define $v = \sqrt{-\mu^2/\lambda}$. We can always write a complex scalar in term of two parts, $\phi = (\phi_a, \phi_b)$, which will form a total of **4** fields since each ϕ_a (or ϕ_b) can be written as,

$$\begin{aligned} \phi_a &= A\left(\phi_1 + i\phi_2\right), \\ \phi_b &= A\left(\phi_3 + i\phi_4\right), \end{aligned} \tag{87}$$

where A is a normalization constant which is found to be $A = 1\sqrt{2}$. The normalization condition for the whole system is then,

$$\langle 0 | \phi | 0 \rangle = v^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2. \tag{88}$$

However, since these are complex scalars, and we have the normalization condition above, we can view this as the constraint equation for a system that has $O(4)$ symmetry. Thus since these are rotations, to view the ground state, we can set three of the "angles" equal to, $\phi_1 = \phi_2 = \phi_4 = 0$, leaving only the ϕ^3 angle (you could choose any angle, but this makes the algebra easier to see).

Exercise 8.5: Show that the proper acceleration perceived by a stationary observer above a Schwarzschild black hole is,

$$A = \frac{GM}{r}\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}, \tag{89}$$

and hence that the Hawking temperature seen at infinity is that of Unruh radiation redshifted from the event horizon.

Solution: The textbook does not give, or go into that much detail on how to find the covariant acceleration. So, from just the principles given in the book, we will find the covariant 4-acceleration.

From the *Principle of Equivalence* we can start in in a local inertial frame, $a^\mu = F^\mu/m$ with $a^\mu = \frac{\partial U^\mu}{\partial \tau}$. Then if we go to a covariant frame, we need something of the form, $a^\mu = \frac{\partial U^\mu}{\partial \tau} + \Gamma^\mu_{\alpha\beta} * (some term)$. To find the "some term" part, we can simply use units to see the covariant 4-acceleration should be,

$$a^\mu = \frac{F^\mu}{m} = \frac{\partial U^\mu}{\partial \tau} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta. \tag{90}$$

The magnitude of the 4-acceleration is then, $a_0^2 = -F^\mu F_\mu/m$ in a give rest frame ($U^\mu = (1, 0, 0, 0)$). Now recall from page 52-53 in equation 2.70, these are the equations of motion for a particle going around a black hole,

$$\begin{aligned} t'\left(1-\frac{2GM}{r^2}\right) &= constant, \\ r^2\phi' &= constant \equiv L. \end{aligned} \tag{91}$$

Then for acceleration, $t''=0$ and $r''=-GM/r^2$ (remember these are the component acceleration equations). Then we simply need to expand these over the square root under equation 90 to give us,

$$a_0=\sqrt{g_{rr}}\frac{GM}{r}=\left(1-\frac{2GM}{r^2}\right)^{-1/2}\left(\frac{GM}{r}\right), \tag{92}$$

(in the text, there is a typo) and we just need to restore the factor of c to get equation 89.The text does not state what the Unruh temperature *is*, but if one has Carroll's text around, then on page 411, the Unruh temperature is simply,

$$T=\frac{a}{2\pi}. \tag{93}$$

At infinity, the r-term will give the Hawking Radiation.

Exercise 8.6: Consider a quantum-mechanical system with a Hamiltonian in the form of a sum over a set of simple harmonic oscillators,

$$H=\sum_{modes}\frac{1}{2}\hbar\omega(aa^\dagger+a^\dagger a). \tag{94}$$

Quantize this in the usual way with the commutator $[a,a^\dagger]=1$, and show that $H=\sum(n+\frac{1}{2})\hbar\omega$. Thus show that the total zero-point energy diverges unless the sum is truncated at some energy. Now quantize with the *anti*commutator $\{a,a^\dagger\}=1$. Prove that $n=0$ or 1, as expected for fermions, and that $H=\sum(n-\frac{1}{2})\hbar\omega$. Hence show that the vacuum energy is exactly zero in a universe with unbroken supersymmetry.

Solution: The commutator version is done in chapter 7 on pages 179-180 (which I will reference a lot). We will also need the information of this chapter from pages 267-269.

Just as we did in chapter 7, we can construct the same number operator $N=a^\dagger a$, but now use the anticommutator relationship. We can follow the same steps as in chapter 7, so first lets compute $Na^\dagger$ and Na,

$$\begin{aligned} Na^\dagger &= (a^\dagger a)a^\dagger = a^\dagger(aa^\dagger) = a^\dagger(1-a^\dagger a) = a^\dagger(1-N), \\ Na &= (a^\dagger a)a = (1-a^\dagger a) = a(1-N). \end{aligned} \tag{95}$$

We can then set the operator states proportional to,

$$\begin{aligned} a^\dagger\left|n\right\rangle &\propto \left|n-1\right\rangle, \\ a\left|n\right\rangle &\propto \left|n-1\right\rangle. \end{aligned} \tag{96}$$

To find their proportionality "constant" we normalize the states,

$$\begin{aligned} \langle an|an\rangle &= 1 = \langle n|\,a^\dagger a\,|n\rangle = n, \\ \left\langle a^\dagger n\middle|a^\dagger n\right\rangle &= 1 = \langle n|\,aa^\dagger\,|n\rangle = 1-n, \end{aligned} \tag{97}$$

which gives the operator equations to be,

$$\begin{aligned} a^\dagger\left|n\right\rangle &= \sqrt{1-n}\left|n-1\right\rangle, \\ a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle. \end{aligned} \tag{98}$$

The only possible values that the modes can take are $n = 0, 1$ which can be seen more clearly by finding the Hamiltonian for this system, which is simply equation 7.10 by instead of plus 1/2 it is minus,

$$H = \sum \frac{\hbar\omega}{2} \left(aa^\dagger - a^\dagger a \right) = \sum \left(N - \frac{1}{2} \right) \hbar\omega, \tag{99}$$

which shows us the vacuum state will always be 0 as long as the *symmetry* of supersymmetry itself is not broken.

9 Chapter 9: The hot big bang

Exercise 9.1: Why is it acceptable to treat a system of charged particles as though they were a non-interacting ideal gas?

Solution: Much of this answer comes from the textbooks answer, but I will fill in detail since their answer seems to come out of nowhere. The best notes I could find to help with this question came from *MIT opencourseware*[4]. We can also take what the book gives us, and make some assumptions from the general scattering procedure for non-relativistic energies from Thomson scattering for a charged particle. So, consider a single two-body collision between an electron and an ion of charge Ze (since any ion will have *some* number of protons Z with an associated charge e). Then from Thomson's formula (also found on *Wiki*),

$$\frac{d\sigma_t}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon_0 mc^2} \right)^2 \left(\frac{1 + cos^2\chi}{2} \right) \rightarrow b^2 * V^4 \frac{d\sigma_t}{d\Omega} = \left(\frac{Ze * e}{4\pi\epsilon_0 \mu c^2} \right)^2, \tag{100}$$

where, to match up with the equation in the book, the cosine portion actually came from integrating over the spatial direction r, which can be viewed as V^2 scaled by some constant called the *impact parameter* b. I also replaced the regular mass with the reduced mass μ. Then, the scattering angle, $d\sigma_t/d\Omega$ for two particles will simply be $tan^2(\theta/2)$ for which if we plug in, divide over the volume, and take a square root, we recover what the text gives us,

$$tan \left(\frac{\theta}{2} \right) = \frac{Ze^2}{4\pi\epsilon_0 \mu V^2 b}. \tag{101}$$

There then must be an effective angle at which scattering will not occur, meaning a proper size of the ion (b_c for critical impact parameter; represents the relative size of a particle),

$$b = \frac{2Ze^2}{12\pi\epsilon_0 kT}, \tag{102}$$

where the kT comes from finding the energy $\langle E \rangle = \langle \mu V^2/2 \rangle = 3kT/2$. From this relationship, we can see that as $T \rightarrow \infty$, the critical impact parameter will go to 0, thus early on in the big bang, we can treat charged particles as non interacting, and this is save as long as T remains large.

Exercise 9.2: Show that the probability for a given state to be occupied is given by the **Gibbs' factor** $p \propto exp[-(E - \mu N)/kT]$, and that the free energy is,

$$F = -kTln(Z_G) + \mu N. \tag{103}$$

Solution: To recall a lot of the thermodynamics used in this chapter, I am using (and recommend) the text, Statistical and Thermal Physics by Gould and Tobochnik (which I will use extensively).

[4]The link for the lecture notes is found here.

One can rexognize that this is the probability of a grand canonical ensemble, and thus we can use the setting of a system with some entropy dS that is surrounded by a heat bath dSb. The total entropy is then,

$$dS_{total} = dS + dS_b = dS + \frac{dE}{T} + \frac{PdV}{T} + \frac{\mu dN}{T} > 0, \tag{104}$$

where the substitution comes from the second law of thermodynamics, $dE_b = TdS_b - PdV_b + \mu dN_b$, and the final equality is from the third law. We can define the availability as,

$$dA = dE + PdV - \mu dN - TdS, \tag{105}$$

which from equation 104 can be used to find the inequality, $dA < 0$. This is a statement that the availability is minimized (which is good). Since this is minimized, there is a corresponding quantity called the *Helmholtz free energy*, which can be found if we hold $dV = dN = 0$ and define a new quantity from equation 105,

$$dA = dE + \cancel{PdV} - \cancel{\mu dN} - TdS = dE - TdS \rightarrow F = E - TS. \tag{106}$$

Now, normally the probability would go as,

$$p_i \propto e^{dS_{tot}/k}, \tag{107}$$

which can be complex to do. However, we can argue that the number of ways to organize the states of the entire system is dominated by the entropy of the heat bath (think of a box of length L embedded into a permeable box of $L + finite\,term$. Then, no matter what the state of the box is in, the heat bath will always have larger states). So, the probability should be,

$$p_i = \alpha e^{dS_b/k} = \alpha e^{-(E_i - \mu N_i)/kT}, \tag{108}$$

where α is some constant, and this is the Gibbs factor.

There is actually a more direct method if we know a few thermodynamic relations and some calculus. We can start with Boltzmann's entropy in terms of probability,

$$\begin{aligned} S &\equiv -k \sum_i p_i ln(p_i) \\ &= k(ln(Z) + \beta\langle E\rangle) \\ &= \frac{\partial}{\partial T}(kTln(Z)) = -\frac{\partial F}{\partial T} \end{aligned} \tag{109}$$

given that $\langle E\rangle = kT^2 \partial ln(Z)/\partial T$. We then can define $F = -kTln(Z_G) + \mu N$ given that we expand the expectation value of the energy. In this method, we are able to skip over finding the Gibbs factor, but this is because it is hidden in the partition function. We can also write this as $F = \langle E\rangle - TS$[5].

Exercise 9.3: Suppose the universe contained only a comoving density n of particles with mass m, which have a lifetime τ and which decay to relativistic products. Calculate the entropy density produced when the particles have finished decaying and compare with the naive answer obtained from assuming instantaneous decay at $t = \tau$.

Solution: There are two methods of completing this problem. First, via units, we can relate the energy density as $u \sim m/a^3 = nm/a^3$ with a being the length (in this case volume) of some space (let it be a cube to keep the algebra easy, and n is unitless). Then, from mimicking page 276, we can get the entropy density,

$$s = \frac{4u}{3T}. \tag{110}$$

[5] For a textbook that goes through this in detail, I was using another text then the one above by Pathira and Beale, Statistical Mechanics.

Now, from equation 9.25. we can relate the energy density to the photon density to get,

$$u_\gamma = \frac{\pi^2}{15}\frac{(kT)^4}{(\hbar c)^3}, \tag{111}$$

which when we plug into the equation above gives,

$$s(\tau) = \frac{1}{a^{9/4}(\tau)}\frac{4kmn}{3}\left(\frac{\pi^2 g_*}{30cm\hbar^3c^3}\right)^{1/4}, \tag{112}$$

where g_* is the degeneracy factor for the photon. However, this is a much more direct way to answer the question. We could simply integrate over the redshifted contributions from the decay products (pg. 293),

$$\begin{aligned} u &= \int (mn)\frac{a(t)e^{-t/\tau}dt}{\tau} \\ &= mna(\tau)\int \tau\frac{e^{-u}}{\tau}du \\ &= \sqrt{\pi}mna(\tau). \end{aligned} \tag{113}$$

So, we were initially off by a factor of $\sqrt{\pi}$.

Exercise 9.4: Show that the general relativistic form of the Boltzmann equation for particles affected by gravitational force and collisions is,

$$\left(p^\mu\frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta\frac{\partial}{\partial p^\mu}\right)f = C. \tag{114}$$

Solution: In general (you could say in flat space), the Boltzmann equation is,

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{force} + \left(\frac{\partial f}{\partial t}\right)_{diffusion} + \left(\frac{\partial f}{\partial t}\right)_{collision}. \tag{115}$$

However, if we are in a cosmological context at high enough temperature, then we can set $df_f = df_c = df_d = 0$, meaning we can equation the above to a constant C. Now, in spacetime, we define our coordinates to some affine parameter λ, and with the chain rule, we get the following,

$$\frac{df}{d\lambda} = \frac{df}{dx^a}\frac{dx^a}{d\lambda} + \frac{df}{dp^a}\frac{dp^a}{d\lambda}, \tag{116}$$

The first term contains a $dx^a/d\lambda = p^a$, while the second term is simple the geodesic equation,

$$\frac{dp^a}{d\lambda} = \frac{d^2x^a}{d\lambda} = -\Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = -\Gamma^a_{bc}p^b p^c, \tag{117}$$

which if we plug-in, and pull out the function (just like when finding the commutator of operators),

$$\left(p^\mu\frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta\frac{\partial}{\partial p^\mu}\right)f = C. \tag{118}$$

Exercise 9.5: Use the following method to derive the relativistic transformation law for specific intensity I_ν. Regard the photons traveling in some range of solid angle $d\Omega$ and frequency range $d\nu$ as a cloud of particles with density $I_\nu d\nu d\Omega$. Derive the transformation laws for particle flux, frequency and solid angle; use these to obtain the transformation law for I_ν and show that I_ν/ν^3 is an invariant.

Solution: We can put ourselves in the (t, x) frame to make things simple. First, lets right down the 4-vectors for a photon with 4-frequency, $k^\mu = (\omega, \mathbf{k}) = (\omega, -\omega cos(\theta))$ and the 4-current, $J^\mu = (\rho, \mathbf{j}) = (n, -ncos(theta))$, where we use the number-particle density n instead of the typical density used for currents ρ. We can transform the 4-frequency, $k^\mu \to k'^\mu$ to get,

$$\begin{aligned} k'^0 &\to \omega' = \omega\gamma(1 + \beta cos(\theta)), \\ k'^j &\to \omega' cos(\theta') = \omega\gamma(cos(\theta) + \beta), \end{aligned} \tag{119}$$

and we get the exact same thing for J'^μ if we switch ω for n. Now, just as the question states, we can find the photon's flux to be, $Id\omega dcos(\theta)$ which is the rate of transport per unit area (a flux). Now suppose the Doppler effect picks up a term when transformed, $\omega' = \alpha\omega$. Then, when we transform the differential,

$$I' d\omega' dcos(\theta') = \alpha^2 I d\omega dcos(\theta). \tag{120}$$

Now plug-in the the Lorentz-transformed factors of ω and $dcos(\theta)$ which allows us to the cancel the side of the equation above to give us, $I' = \alpha^3 I$, showing that I/ω^3 is invariant.

Exercise 9.6: Show that the momentum-space integrals in the expression for the neutron decay rate reduce to the integral,

$$\int_{m_e}^{Q} p_e \epsilon_e \left(Q - \epsilon_e\right)^2 d\epsilon_e = 1.636 m_e^5. \tag{121}$$

Solution: Our starting point can be equation 9.69 to reference from. The differential in equation 9.69, $d^3 p_\nu$ gives,

$$d^3 p_\eta = 4\pi p_\nu^2 dp, \tag{122}$$

if we integrate over a spherical region of space. This would reduce the integral in 9.69 to be,

$$4\pi \int \delta(\epsilon_\eta + \epsilon_e - Q) p_\nu dp = 2\pi (Q - \epsilon_e)^2. \tag{123}$$

Now, to deal with the electron-momentum differential $d^3 p_e$, consider $\epsilon_e^2 = m_e^2 + p_e^2$, which if we take the differential (the trick is to split the the squared energy density and momentum) we get, $\epsilon_e d\epsilon_e = p_e dp_e$. We can then substitute this into the original integral of 9.69 and what we found previously to get,

$$\int_{m_e}^{Q} p_e \epsilon_e (Q - \epsilon_e)^2 d\epsilon_e \sim 2 m_e^5. \tag{124}$$

If you do it explicitly, when using $Q = 2.531 m_e$, you should find the exact value (I simply plugged in $m_e \sim Q$, but do this after expanding the polynomial).

10 Chapter 10: Topological defects

Exercise 10.1: Show that the light deflection angle produced by a cosmic string at and angle θ to the plane of the sky is smaller than the deflection that arises when the string lies in the plane of the sky:

$$\alpha = 4\pi \frac{G\mu}{c^2} cos(\theta). \tag{125}$$

Solution: Consider a cosmic string that lies in the z-axis but has its deficit angle, $\epsilon = 8\pi G\mu$ with respect to the x-axis. Now, picture two photons that are perpendicular to the ends of the string that pass by it. One photon remains the same, but the second, with initial momentum, $\mathbf{p}_{i,1} = (x, y, z) = (0, pcos(\theta), psin(\theta))$. The cross section of the string's end and the momentum becomes,

$$\mathbf{p}_{i,1} \wedge \mathbf{A}_{string} = -8\pi G\mu p_{1,y} \hat{x}. \tag{126}$$

Now, this shows that the final momentum of the two photons is perpendicular to the initial pair,

$$(\mathbf{p_1} \wedge \mathbf{p_2})_f =\perp (\mathbf{p}_{1,i} \wedge \mathbf{p}_{i,2})_f, \tag{127}$$

with magnitude,

$$|(\mathbf{p}_{1,i} \wedge \mathbf{p}_{i,2})_f| = 8\pi G cos(\theta). \tag{128}$$

Exercise 10.2: Show that gravitational radiation by a distorted oscillating loop of cosmic string will produce an emitted power $\sim G\mu^2$, and hence that the smallest surviving loops will have sizes $\sim G\mu t$.

Solution: Recall from chapter 2.3, or form chapter 36 from Gravitation, that the quadrupole radiation has a power-formula,

$$-\dot{E} = \frac{G}{5}\langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle, \tag{129}$$

the reduced quadrupole-moment tensor goes such as, $I_{ij} \sim ML^2$, and the mass of the cosmic string goes as, $M \sim 2\pi\mu L$. We also know that the frequency (since the string travels at speed c), $\omega \sim 1/L$. Then, from the text, the radiation of the cosmic string's gravitational energy should go as (eq. 2.49),

$$-\dot{E} \sim GM^2L^4\omega^6 \sim GM^2L^4\frac{1}{L^6} \sim \frac{M^2}{L^2} \sim G\mu^2, \tag{130}$$

(if we drop the factor of two and π). Now, if we let $\dot{E} \sim -\dot{M}c^2$, then integrate with respect to ΔT and Δt (after we plug in what we know about M) we find,

$$L - L_i \sim G\mu(t - t_i). \tag{131}$$

11 Chapter 11: Inflationary cosmology

Exercise 11.1: Verify that the potential,

$$V(\phi) \propto exp\left(\sqrt{\frac{16\pi}{pm_p^2}}\phi\right) \tag{132}$$

leads to inflation with a scale-factor dependence $a(t) \propto t^p$. Since this is not de Sitter space, why does the resulting behavior still lead to zero comoving spatial curvature?

Solution: From equation 11.21, we have the general equation of motion for a scalar being,

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + \frac{dV}{d\phi} = 0. \tag{133}$$

Based on the assumption of page 332, we can reduce equation 133 to just a first derivative term (keep this in mind).

The Friedmann equation is,

$$H^2m_p^2 = \frac{8\pi}{3}\left(V + \frac{\dot{\phi}^2}{2}\right) - \frac{km_p^2}{R^2}, \tag{134}$$

where $H = \dot{R}/R$ and k is the curvature constant. If we assume the universe is flat, $k = 0$, we have to check that V does not decline quickly. Suppose that the potential term in-between the parentheses will being a constant multiple of the potential, $\dot{\phi}^2/2 = \alpha V$. Then the evolution equation becomes,

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \tag{135}$$

We can manipulate the $\ddot{\phi}$ term into,

$$\ddot{\phi} = \dot{\phi}\frac{d\dot{\phi}}{d\phi} = (\dot{\phi}^2/2)' = \alpha V. \tag{136}$$

We can do the same for $\dot{\phi} = -\sqrt{2\alpha V}$ (we can choose the sign in an arbitrary manner here with the purpose to have the scalar field roll down the potential towards the origin, as seen in figure 11.2). The evolution equation becomes,

$$(1+\alpha)V' = \sqrt{48\pi\alpha(1+\alpha/m_p^2)}V \to V \propto exp\left(\sqrt{\frac{16\pi}{pm_p^2}}\phi\right). \tag{137}$$

Now, if we plug these into the equations of state given in chapter 2, we find a power term of $p = (1+\alpha)/3\alpha$, and with $V \propto H^2$ gives $V(t) \propto R^{-2/p}$. Thus we would require a slow-rolling parameter of $\epsilon = p/2$.

Exercise 11.2: For the effective potential $V = aT^2|\phi|^2 - b|\phi|^3 + \lambda|\phi|^4$, obtain the critical temperature below which there are two minima, and the critical temperature for which these are of equal V. How high is the energy barrier around the origin as a function of T?

Solution: Consider the potential given when $b = \lambda = 1$ which can be written into the potential if we define $y \equiv \lambda\phi/b$ which gives,

$$y^4 - y^3 + \frac{aT^2\lambda}{b^2}y^2 = \left(\frac{\lambda}{b}\right)^4. \tag{138}$$

If we define $A \equiv aT^2\lambda/b^2$, and putting everything to one side so we can solve $V' - 0$, then just like any third degree polynomial, we get $y = 0$ and $y = (3 \pm \sqrt{9-32A})/8$, with the positive root being the minimum. Now solving for $V = 0$ gives $y = [1 \pm \sqrt{1-4A}]/2$, which at the degenerate point, the second minimum is at $V = 0$ giving $A_2 = 1/4$. Putting the expression we defined earlier at the local maximum (plug in the A value) gives $y_{max} \simeq 2A/3$.

Exercise 11.3: Consider the scalar field at a given point in the inflationary universe. Each e-folding of the expansion produces new classical fluctuations, which add incoherently to those previously present. Show that, if the field is sufficiently far from the origin in a polynomial potential, these fluctuations produce a random walk of $\phi(t)$ that overwhelms the classical trajectory in which ϕ tries to roll down the potential. Cast the behavior as a diffusion process, and thus argue that some parts of the universe will reach infinite ϕ, thus inflating eternally.

Solution: We can start with page 339 equation 11.38, with the classical amplitude-fluctuations,

$$\delta\dot{\phi} = \frac{H^2}{2\pi}, \tag{139}$$

with a new disturbance every $\Delta t \sim 1/H$. The slow role equation (pg. 332), $3H\dot{\phi} = -dV/d\phi = V'$ gives us a trajectory of ϕ, and with $3m_p^2H^2 = 8\pi V$, we can set $\dot{\phi}$ to be,

$$\dot{\phi} = \frac{\Delta\phi}{\Delta t} => \Delta\phi = \frac{-V'}{3H^2} = \frac{-V'}{V}\frac{m_p^2}{8\pi}. \tag{140}$$

Consider now a general potential,

$$V(\phi) = \frac{\lambda|\phi|^n}{nm_p^{n-4}}. \tag{141}$$

If ϕ is small, the trajectory of the roll is still classical, but at large ϕ the motion is stochastic (pg. 341) , which will have a corresponding probability distribution if we define the random walk as with the steps being the e-folding governed by $N = H\Delta t$ to give, $\Delta\phi = \sqrt{N}H/2\pi$. $P(\phi)$ is then, as told in the question, a diffusion equation,

$$\frac{\partial P(\phi;t)}{\partial t} = \frac{\partial^2(DP(\phi;t))}{d\phi^2}, \tag{142}$$

if we left the diffusion coefficient be related to the rms, $x_{rms} = \sqrt{2Dt}$. Then, only have of the space is driven to infinite values of ϕ.

12 Chapter 12, 13, 14, 15, 16, and 17.

As of right now, the chapters listed in the section header are not of interests, although the more I dig into the interlinking of cosmology plus quantum gravity, I may learned more in-depth on how measurements and predictions are formulated. We go into chapter 18 since I know that the CMB is of great importance for quantum gravity predictions.

13 Chapter 18: Cosmic background fluctuations

Exercise 18.1: In terms of the 2D power spectrum of the temperature fluctuations on the sky, $\Delta^2(K)$, we are often interested in the moments,

$$\sigma_n^2 = \int_0^\infty \mathcal{T}^2(K)K^{2n}\frac{dK}{K}. \tag{143}$$

From this definition, show that the behavior of the temperature autocorrelation function near the origin is

$$\frac{C(\theta)}{\sigma_0^2} \simeq 1 - \frac{\gamma^2}{2}\left(\frac{\theta}{\theta^*}\right)^2 + \frac{\gamma^2}{16}\left(\frac{\theta}{\theta^*}\right)^4 + ..., \tag{144}$$

and give the combination of moments that define the parameters γ and θ^*.

Solution: The moments can be put in terms of derivatives of the correlation function,

$$C(\theta) = \int_0^\infty \Delta^2(K)J_0(K\theta)\frac{dK}{K}, \tag{145}$$

(which is equation 18.5). If we take the derivative of equation 145 with respect to J_0 will give,

$$\sigma_n^2 = (-1)^n\frac{2^{2n}(n!)^2}{(2n)!}C^{2n}(0). \tag{146}$$

We can contruct BBKS parameters[6],

$$\theta^* \equiv \sqrt{2}\frac{\sigma_1}{\sigma_2},\ and\ \gamma \equiv \frac{\sigma_1^2}{\sigma_0\sigma_2}, \tag{147}$$

which can be plugged into the derivative above, and find the requested expansion of equation 144.

Exercise 18.2: Using the projection equation for the 2D temperature power spectrum as an integral over the 3D spectrum, obtain the corresponding integral that gives the correlation function $C(\theta)$ in terms of $\mathcal{T}_{3D}^2$.

Solution: This is referencing to page 590. We can Fourier-transform equation 18.11, or we can find the temperature correlation function with equation 18.7 instead by introducing polar coordinates and perform the integral over k-space,

$$C(\Delta\psi) = \frac{1}{2}\int_0^\infty\int_{-1}^1 \mathcal{T}^2(k)J_0\left(kR_H\Delta\psi\right)e^{-k^2\sigma_r^2\mu^2}\frac{dkd\mu}{k}. \tag{148}$$

[6] For reference on where they come from, since the text does a partial job of this, the original paper is here,

References

[1] John A. Peacock, *Cosmological Physics.* (Cambridge University Press, 1999).

[2] Riley, Hobson, and Bence, *Mathematical Methods for Physics and Engineering,* 3^{rd} *edition.* (Cambridge University Press, 2006).

[3] Lightman, Press, Price, and Teukolsky, *Problem Book in Relativity and Gravitation.* (Princeton University Press; First Edition, 1975).